Circulant preconditioners from B-splines

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ABSTRACT

In this paper, we propose a new family of circulant preconditioners for solving Toeplitz systems. They are based on B-splines. The R. Chan and T. Chan preconditioners can be constructed from the first and the second order B-splines. Numerical results show that preconditioners from higher-order B-splines perform much better than well-known ones even in the cases where the Toeplitz matrices are ill-conditioned. Like that of the other circulant preconditioners, the construction of B-spline preconditioners requires only the entries of the given Toeplitz matrix and does not require an apriori knowledge of its generating function. Thus they are most suitable for applications where the generating function of the given Toeplitz matrix is not known explicitly.

Keywords: B-spline, Preconditioner, Toeplitz matrix, Circulant matrix

1. INTRODUCTION

An n-by-n matrix A_n is said to be *Toeplitz* if it is constant along its diagonals:

$$A_n = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{2-n} & a_{1-n} \\ a_1 & a_0 & a_{-1} & & a_{2-n} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix}, \tag{1}$$

i.e., if $A_n = [a_{ij}]$, then $a_{ij} = a_{i-j}$ for all $i, j = 0, \dots n-1$. In this paper, we are interested in solving Toeplitz system $A_n\mathbf{x} = \mathbf{b}$.

Toeplitz systems arise in a variety of applications in mathematics and engineering. A straightforward application of the Gaussian elimination method will result in an algorithm of $O(n^3)$ complexity. An $O(n^2)$ method, related to signal processing techniques, was invented by Levinson[8] in 1946. Around 1980, superfast direct solvers of complexity $O(n \log^2 n)$ were developed, see for instance, Brent, Gustavson, and Yun.[1] Preconditioned conjugate gradient methods with circulant preconditioners were developed in the mid-80s. The complexity is $O(n \log n)$ for large classes of Toeplitz systems, see the survey paper by Chan and Ng.[4]

The main aim of this paper is to propose a family of circulant preconditioners from B-splines. Using this approach, the R. Chan and T. Chan preconditioners can be constructed from the first and the second order B-splines respectively. We observe that the kernels corresponding to these B-spline preconditioners are closer to the Dirac delta function when the order of the B-spline is higher. Our numerical results show that preconditioners from higher-order B-splines perform much better than well-known ones even in cases where the Toeplitz matrices are ill-conditioned. Like that of the other circulant preconditioners, the construction of B-spline preconditioners requires only the entries

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of the given Toeplitz matrix and does not require an apriori knowledge of its generating function. Thus they are most suitable for applications where the generating function of the given Toeplitz matrix is not known explicitly.

The outline of the paper is as follows. In §2, we introduce some well-known circulant preconditioners and recall an earlier result that links these preconditioners to some well-known kernels. In §3, we introduce a translated and scaled version of standard B-splines and give the procedure of constructing circulant preconditioners from these translated and scaled B-splines. In §4, we test the B-spline preconditioners on different classes of Toeplitz matrices and on a Toeplitz least squares problem.

2. CIRCULANT PRECONDITIONERS FROM KERNELS

Let f be a function defined on $[-\pi, \pi]$. Define

$$a_k \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

the Fourier coefficients of f. For all $n \geq 1$, let A_n be the n-by-n Toeplitz matrix with entries $a_{j,k} = a_{j-k}$, $0 \leq j, k < n$. The function f is called the *generating function* of the sequence of Toeplitz matrices A_n, $n \geq 1$. We will write $A_n = A_n[f]$ for clarity.

Given $A_n[f]$, the Strang preconditioner[2] $S_n[f]$ is defined to be the circulant matrix that copies the central diagonals of $A_n[f]$ and reflects them around to complete the circulant. More precisely, the kth entry in the first column of $S_n[f]$ is given by

$$(S_n[f])_{k0} = \begin{cases} a_k, & 0 \leq k \leq \lfloor n/2 \rfloor, \\ a_{k-n}, & \lfloor n/2 \rfloor < k < n. \end{cases}$$

T. Chan's preconditioner[6] $T_n[f]$ is defined to be the circulant matrix with diagonals that are arithmetic averages of the diagonals of $A_n[f]$ (extended to length n by wrap-around when necessary). More precisely, the entries in the first column of $T_n[f]$ are given by

$$(T_n[f])_{k0} = \frac{1}{n}\{(n-k)a_k + k\bar{a}_{n-k}\}, \quad 0 \leq k < n.$$

R. Chan's circulant preconditioner[2] $R_n[f]$ has the first column given by

$$(R_n[f])_{k0} = \begin{cases} a_0, & k = 0, \\ a_k + \bar{a}_{n-k}, & 0 < k < n. \end{cases}$$

A unifying approach of constructing circulant preconditioners is given in R. Chan and Yeung,[5] where it is shown that the above circulant preconditioners can be derived by using the convolution products of the generating function f with some well-known kernels such as the Dirichlet and the Fejér kernels. In fact, the eigenvalues of Strang's preconditioner $S_n[f]$ are given by

$$\lambda_j(S_n[f]) = (\hat{D}_{\lfloor \frac{n}{2} \rfloor} * f)(\frac{2\pi j}{n}), \quad 0 \leq j < n,$$

where

$$(g * f)(\theta) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta - \phi)f(\phi)d\phi$$

and

$$\hat{D}_k(\theta) = \frac{\sin(k + \frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \quad k = 1, 2, \ldots,$$

is the well-known Dirichlet kernel. The eigenvalues of T. Chan's preconditioner $T_n[f]$ are given by

$$\lambda_j(T_n[f]) = (\hat{F}_n * f)(\frac{2\pi j}{n}), \quad 0 \leq j < n, \tag{3}$$

where

$$\hat{F}_k(\theta) = \frac{1}{k}\left(\frac{\sin\frac{k}{2}\theta}{\sin\frac{1}{2}\theta}\right)^2, \quad k = 1, 2, \ldots,$$

is the Fejér kernel. The eigenvalues of R. Chan's preconditioner $R_n[f]$ are given by

$$\lambda_j(R_n[f]) = (\hat{D}_{n-1} * f)(\frac{2\pi j}{n}), \quad 0 \le j < n. \tag{4}$$

The idea can be used to design circulant preconditioners from other kernels $\hat{C}_n$. More precisely, we can construct circulant preconditioners $C_n[f]$ where the eigenvalues are given by

$$\lambda_j(C_n[f]) = (\hat{C}_n * f)(\frac{2\pi j}{n}), \quad 0 \le j < n. \tag{5}$$

In Chan and Yeung,[5] several circulant preconditioners were constructed using this approach. Moreover, it was proved that when f is a 2π-periodic continuous positive function, then a sufficient condition for $C_n[f]$ to be a good preconditioner (in the sense that the preconditioned system $C_n^{-1}[f]A_n[f]$ converges superlinearly) is that $\lim_{n\to 0} \|\hat{C}_n * f - f\|_\infty = 0$.

Notice that $\hat{C}_n * f$ is just a smooth approximation of the generating function f itself. According to (5), the circulant preconditioner $C_n[f]$ is designed so that its eigenvalues approximate the values of f at $2\pi j/n$, $0 \le j < n$. Thus if $f(2\pi j/n)$ can be computed efficiently, then the circulant preconditioners with eigenvalues given by $f(2\pi j/n)$ is certainly a good choice. The corresponding kernel is the Dirac delta function δ as $\delta * f = f$. In other words, if $\hat{C}_n$ is close to δ, then the constructed circulant matrix is expected to be a good preconditioner, at least for the class of 2π-periodic continuous positive functions. Unfortunately, in most applications, we are only given one Toeplitz system to solve, and therefore we do not have an apriori knowledge of f. In this case, one can only approximate the values $f(2\pi j/n)$ as closely as possible as is done in (3) and (4).

3. CIRCULANT PRECONDITIONERS FROM B-SPLINES

For all $j \ge 0$, define the truncated power function as:

$$(x-y)_+^j = \begin{cases} 0, & x < y, \\ (x-y)^j, & x \ge y. \end{cases}$$

For $m = 1, 2, \ldots$, the m-th order B-spline, associated with simple knots at $x = 0, 1, \cdots, m$, is defined as

$$Q^m(x) = \sum_{i=0}^{m} \frac{(-1)^i \begin{pmatrix} m \\ i \end{pmatrix} (x-i)_+^{m-1}}{m!}.$$

We note that $Q^m(x) \in C^{m-2}(\mathbb{R})$ and that there is a recursive relationship in generating $Q^m(x)$:

$$Q^m(x) = \frac{xQ^{m-1}(x) + (m-x)Q^{m-1}(x-1)}{m}, \quad m \ge 2,$$

see Schumaker.[9]

In order to construct our preconditioners later on, we introduce a translated and scaled version of $Q^m(x)$:

$$B^m(x) = \frac{1}{c_m}\left[Q^m\left(x + \frac{m}{2}\right)\right], \quad x \in \mathbb{R}.$$

where c_m is a normalization constant so as to make $B^m(0) = 1$. This family of splines is symmetric about the origin and has support on $[-m/2, m/2]$. For even m's, it has simple knots at the integers, while for odd m's, the knots are at the mid-points between the integers. As examples, let us give the formula for $B^m(x)$ for $m = 1, 2$ and 3 below. Their graphs are given in Figure 1.

$$B^1(x) = \begin{cases} 1, & -1/2 \leq x < 1/2, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

$$B^2(x) = \begin{cases} 1+x, & -1 \leq x < 0, \\ 1-x, & 0 \leq x < 1. \end{cases} \tag{7}$$

$$B^3(x) = \begin{cases} \dfrac{2x^2}{3} + 2x + \dfrac{3}{2}, & -3/2 \leq x < -1/2, \\[2ex] \dfrac{-4x^2}{3} + 1, & -1/2 \leq x < 1/2, \\[2ex] \dfrac{2x^2}{3} - 2x + \dfrac{3}{2}, & 1/2 \leq x < 3/2. \end{cases}$$

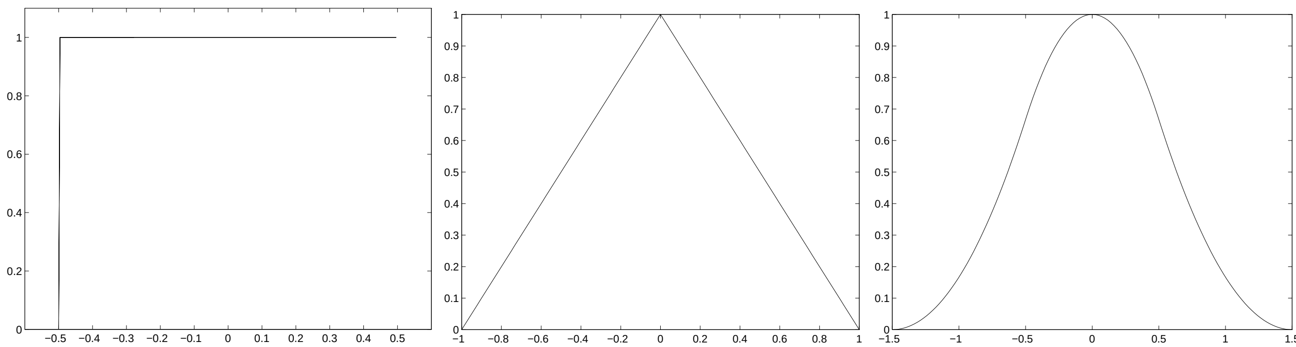

Figure 1. $B^1(x)$ (left), $B^2(x)$ (center) and $B^3(x)$ (right).

Now we propose a numerical procedure for constructing circulant preconditioners from $B^m(x)$. We only need to construct the first columns of the circulant preconditioners as they uniquely define the matrices. Define

$$b_k^m = B^m\left(\frac{mk}{2n}\right), \quad -n < k < n, \quad m = 1, 2, \ldots. \tag{8}$$

By (6), we have $b_k^1 = 1$ for $-n < k < n$. By (7), we see that

$$b_k^2 = \begin{cases} \dfrac{n+k}{n}, & -n < k < 0, \\ \dfrac{n-k}{n}, & 0 \leq k < n. \end{cases} \tag{9}$$

Using (4), one can show that the eigenvalues of $R_n[f]$ are given by

$$\lambda_j(R_n[f]) = (f * \hat{D}_{n-1})(\frac{2\pi j}{n}) = a_0 + \sum_{k=1}^{n-1} a_k \zeta_j^k + \sum_{k=1}^{n-1} \bar{a}_k \bar{\zeta}_j^k = a_0 + \sum_{k=1}^{n-1} \{a_k + \bar{a}_{n-k}\} \zeta_j^k,$$

where $\zeta_j = e^{2\pi i j/n}$, see Chan and Yeung[5] for details. Hence in terms of b_k^1, which are constantly 1 for $|k| < n$, we have

$$\lambda_j(R_n[f]) = a_0 + \sum_{k=1}^{n-1} \{b_k^1 a_k + b_{-(n-k)}^1 \bar{a}_{n-k}\} \zeta_j^k.$$

Thus the first column of R. Chan's circulant preconditioner can be written as

$$(R_n[f])_{k0} = b_k^1 a_k + b_{-(n-k)}^1 \bar{a}_{n-k}, \quad 0 \leq k < n.$$

For T. Chan's preconditioner $T_n[f]$, its eigenvalues are given by (cf (3))

$$\lambda_j(T_n[f]) = (f * \hat{F}_n)(\frac{2\pi j}{n}) = \frac{1}{n}\sum_{k=-(n-1)}^{n-1}(n-|k|)a_k\zeta_j^k = \sum_{k=0}^{n-1}\frac{n-k}{n}a_k\zeta_j^k + \sum_{k=1}^{n-1}\frac{k}{n}\bar{a}_{n-k}\zeta_j^k.$$

Hence in terms of b_k^2 given in (9), we have

$$\lambda_j(T_n[f]) = \sum_{k=0}^{n-1} b_k^2 a_k \zeta_j^k + \sum_{k=1}^{n-1} b_{-(n-k)}^2 \bar{a}_{n-k}\zeta_j^k.$$

Correspondingly, the first column of T. Chan's circulant preconditioner is given by

$$(T_n[f])_{k0} = b_k^2 a_k + b_{-(n-k)}^2 \bar{a}_{n-k}, \quad 0 \le k < n.$$

For higher order B-spline $B^m(x)$, we can define the circulant preconditioner accordingly:

$$(B_n^m[f])_{k0} = b_k^m a_k + b_{-(n-k)}^m \bar{a}_{n-k}, \quad 0 \le k < n, \quad m = 1, 2, \ldots, \tag{10}$$

where b_k^m are given in (8). Its eigenvalues are given by

$$\lambda_j(B_n^m[f]) = \sum_{k=0}^{n-1} b_k^m a_k \zeta_j^k + \sum_{k=1}^{n-1} b_{-(n-k)}^m \bar{a}_{n-k}\zeta_j^k, \quad 0 \le j < n.$$

We emphasize that the construction of the first column of $B_n^m[f]$ requires no apriori knowledge of the generating function f. All we need are the entries a_j of the given Toeplitz matrix, see (10).

Using the relationship (5), $B_n^m[f]$ can be considered as circulant preconditioners constructed from the kernels

$$\hat{B}_n^m(\theta) \equiv \sum_{k=-n}^{n} b_k^m e^{ik\theta}, \quad \theta \in [-\pi, \pi].$$

Recall from the discussion in §2 that if $\hat{B}_n^m(\theta)$ is closer to the Dirac delta function δ, we expect better behaviour of the corresponding preconditioners. In Figures 2–3 below, we plot the function $\hat{B}_n^1(\theta)$ for $n = 32, 64, 128$ and 256. We see that as $n \to \infty$, the function approaches to δ. However, we still see ripples near the origin when $n = 256$. In Figures 4–5, we plot the functions $\hat{B}_{64}^m(\theta)$ for $m = 1, 2, 3$ and 4. It is clear that $\hat{B}_{64}^m(\theta)$ tends to δ much faster for larger m. The same conclusion holds for all other n we have tried.

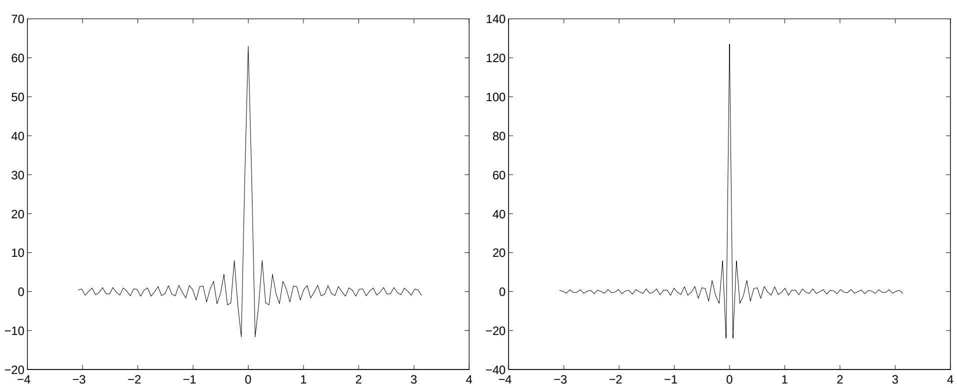

Figure 2. $\hat{B}_n^1(\theta)$ for n=32 (left) and 64 (right).

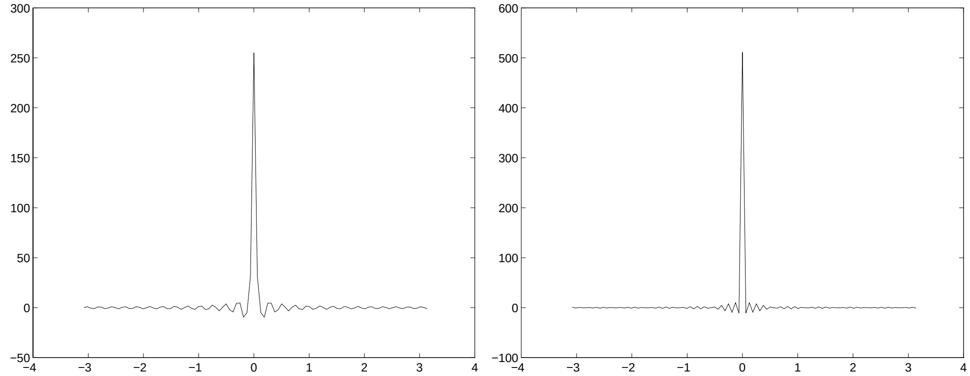

Figure 3. $\hat{B}_n^1(\theta)$ for n=128 (left) and 256 (right).

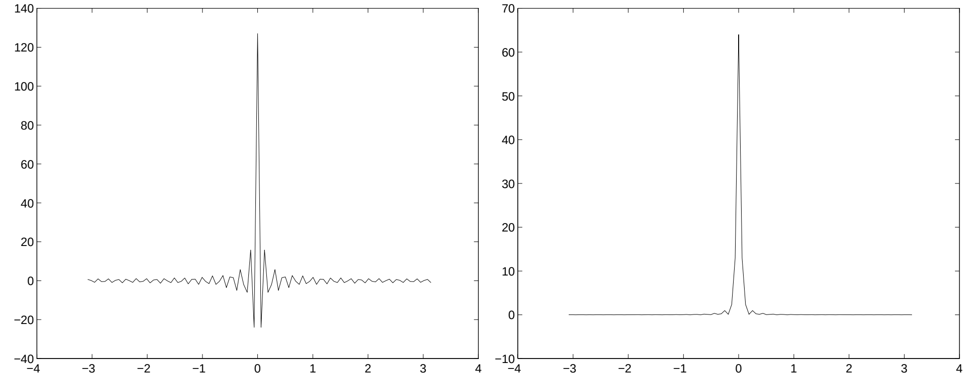

Figure 4. $\hat{B}_{64}^m(\theta)$ for m=1 (left) and 2 (right).

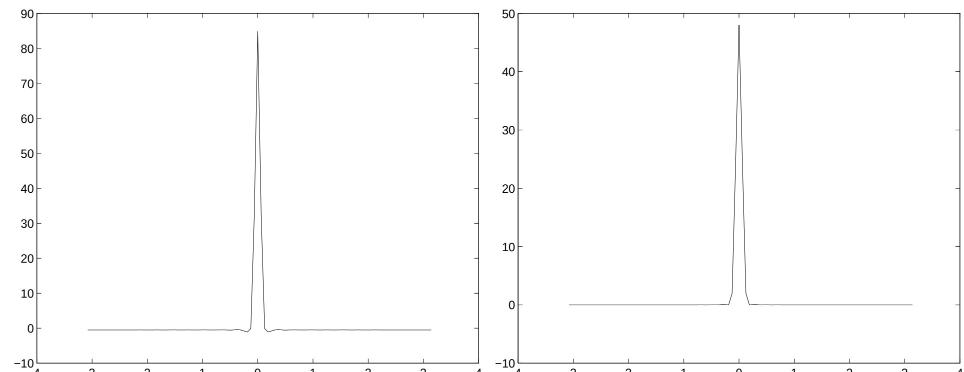

Figure 5. $\hat{B}_{64}^m(\theta)$ for m=3 (left) and 4 (right).

4. NUMERICAL RESULTS

In this section, we compare the B-spline circulant preconditioners with other preconditioners. We first test their performance on fourteen functions defined on $[-\pi, \pi]$ that come from four different classes.

I. Positive generating functions with no jumps:

- $f_1(\theta) = \theta^4 + 1$.
- $f_2(\theta) = |\theta|^3 + 0.01$.

II. Positive generating functions with jumps:

- $f_3(\theta) = (\theta+\pi)^2 + 1$,
- $f_4(\theta) = \begin{cases} \dfrac{0.9\theta}{\pi} + 10, & -\pi < \theta \le 0, \\ \\ \dfrac{0.9\theta}{\pi} + 0.1, & 0 < \theta \le \pi. \end{cases}$

III. Non-negative generating functions with no jumps:

- $f_5(\theta) = \theta^2$,
- $f_6(\theta) = \theta^4$,
- $f_7(\theta) = |\theta|^3$,
- $f_8(\theta) = 1 - \cos\theta$.
- $f_9(\theta) = \theta^2|\theta^2 - 1|$,
- $f_{10}(\theta) = \pi^2\theta^2 - \theta^4$,

IV. Non-negative generating functions with jumps:

- $f_{11}(\theta) = \begin{cases} \theta^2, & |\theta| \le \pi/2, \\ 1, & \pi/2 < |\theta| < \pi, \end{cases}$
- $f_{12}(\theta) = \begin{cases} \theta^2, & \theta \le 0, \\ \theta, & \theta > 0, \end{cases}$
- $f_{13}(\theta) = |\theta(\theta+1)|$,
- $f_{14}(\theta) = (\theta+\pi)^2$.

The Toeplitz matrices $A_n[f]$ are formed by evaluating the Fourier coefficients of the test functions, see (2) and (1) and the systems are solved by the preconditioned conjugate gradient method with different preconditioners. In our tests, the vector of all ones is the right-hand side vector, the zero vector is the initial guess, and the stopping criterion is $\|r_q\|_2/\|r_0\|_2 \le 10^{-7}$, where r_q is the residual vector after the q iterations. All computations are done by matlab on an IBM 43P workstation.

The following tables show the numbers of iterations required for convergence for different choices of preconditioners. In the tables, I denotes no preconditioner, S the Strang preconditioner, and B_n^m are the B-spline preconditioners for $m = 1,2,3,4,5$ and 6. We recall that B_n^1 and B_n^2 are respectively the R. Chan and T. Chan circulant preconditioners, see (4) and (3) respectively. The symbol BT for functions f_5 and f_6 denotes the band-Toeplitz preconditioners.[3] They are only defined for Toeplitz matrices generated by even functions with zeros of even order. Also apriori knowledge of the generating function is required and hence they may not be practical in applications where only one system is given, see for example the least squares Toeplitz system we will consider at the end of this section. For f_8, the preconditioners S and B_n^1 are singular and hence are not used. The numerical results show that the performances of higher-order B-spline preconditioners are the best in general, and for all the tests we tried, B_n^3 is already a good choice.

	f_1							f_2						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	8	20	37	56	67	70	71	8	21	52	135	281	398	434
S	8	8	6	5	5	5	5	8	10	11	10	8	7	7
B_n^1	6	5	5	5	5	5	5	7	8	9	8	7	7	7
B_n^2	8	7	7	6	6	6	5	8	13	15	19	15	12	10
B_n^3	6	5	5	5	5	5	5	8	8	8	7	7	7	7
B_n^4	7	6	5	5	5	5	5	8	9	8	7	7	7	7
B_n^5	6	6	5	5	5	5	5	8	9	8	7	7	7	7
B_n^6	7	6	5	5	5	5	5	9	9	8	7	7	7	7

	f_3							f_4						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	16	31	38	40	39	38	38	16	30	40	44	48	50	52
S	10	14	17	19	19	19	19	16	20	23	27	31	39	40
B_n^1	10	11	12	11	12	12	12	11	13	15	15	17	17	19
B_n^2	10	11	11	12	11	12	12	11	13	15	15	17	17	19
B_n^3	7	8	9	9	9	10	10	9	12	13	14	14	14	15
B_n^4	8	8	9	9	10	10	11	10	11	13	14	15	15	16
B_n^5	8	8	9	9	10	10	11	10	11	13	14	15	16	17
B_n^6	8	9	9	10	10	10	11	10	12	14	15	15	16	17

	f_5							f_6						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	8	17	38	83	178	374	770	9	31	113	544	>1000	>1000	>1000
S	7	7	7	7	8	8	8	8	14	21	36	121	406	>1000
B_n^1	5	7	7	7	7	7	7	9	12	18	32	79	657	>1000
B_n^2	8	10	12	14	18	22	28	9	16	26	65	177	484	>1000
B_n^3	6	6	8	8	8	8	8	9	12	13	15	22	30	49
B_n^4	7	7	8	8	8	9	9	9	12	15	18	23	39	68
B_n^5	7	7	8	8	9	9	9	9	12	15	17	21	31	48
B_n^6	7	7	8	9	9	9	9	9	12	15	17	22	30	55
BT	8	10	11	12	12	12	12	8	15	20	24	27	29	30

	f_7							f_8						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	8	22	61	179	621	>1000	>1000	8	16	32	32	64	256	512
S	8	10	13	16	20	39	75	*	*	*	*	*	*	*
B_n^1	8	10	10	13	20	27	42	*	*	*	*	*	*	*
B_n^2	8	13	17	25	37	101	198	7	8	10	13	15	19	25
B_n^3	8	9	10	10	13	14	15	6	6	6	7	7	7	7
B_n^4	8	10	10	11	13	15	16	6	6	6	8	8	8	8
B_n^5	8	10	11	11	13	15	16	6	6	7	8	8	8	8
B_n^6	9	10	10	11	14	15	16	6	7	7	8	8	8	8

	f_9							f_{10}						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	10	30	90	252	600	>1000	>1000	8	16	32	62	118	225	436
S	8	15	21	23	25	22	31	7	7	9	9	9	10	11
B_n^1	9	15	17	15	17	22	19	6	6	8	8	8	8	11
B_n^2	9	16	22	27	32	40	58	7	9	10	13	15	20	24
B_n^3	9	10	10	12	14	15	15	7	7	8	8	9	9	10
B_n^4	9	11	11	12	14	16	17	7	7	9	9	9	9	12
B_n^5	9	11	11	13	14	16	16	7	8	9	9	9	9	12
B_n^6	9	11	11	13	14	15	16	8	8	9	9	9	10	12

n	f_{11} 16	32	64	128	256	512	1024	f_{12} 16	32	64	128	256	512	1024
I	7	11	20	41	86	178	372	16	36	71	134	257	486	929
S	7	9	11	14	18	22	26	11	11	12	13	14	15	18
B_n^1	7	9	11	12	15	21	25	10	11	11	12	13	15	18
B_n^2	7	9	11	13	15	19	23	10	12	14	16	19	23	30
B_n^3	7	8	9	10	10	10	12	8	9	11	12	13	15	17
B_n^4	7	8	10	10	10	11	12	9	10	11	12	13	15	16
B_n^5	7	8	10	10	11	11	12	9	10	12	12	13	15	16
B_n^6	7	8	10	10	11	11	12	9	10	12	13	14	16	17

n	f_{13} 16	32	64	128	256	512	1024	f_{14} 16	32	64	128	256	512	1024
I	8	17	38	83	178	374	770	16	34	76	167	352	678	>1000
S	7	7	7	7	8	8	8	11	16	26	42	85	203	755
B_n^1	5	7	7	7	7	7	7	11	15	19	28	43	67	113
B_n^2	8	10	12	14	18	22	28	11	15	19	27	41	66	111
B_n^3	6	6	8	8	8	8	8	8	10	12	14	19	30	41
B_n^4	7	7	8	8	8	9	9	9	10	12	12	20	25	34
B_n^5	7	7	8	8	9	9	9	9	9	12	14	18	24	34
B_n^6	7	7	8	9	9	9	9	9	10	12	14	18	21	27

Next we consider a Toeplitz least square problem. In this example, we construct a 100×100 Toeplitz matrix A with (i,j) entries given by

$$a_{ij} = \begin{cases} 0, & \text{if } |i-j| > 8, \\ \frac{4}{51} g(0.15, \frac{4}{51}(i-j)), & \text{otherwise,} \end{cases}$$

where

$$g(\sigma,\gamma) = \frac{1}{2\sqrt{\pi}\sigma} \exp(-\frac{\gamma^2}{4\sigma^2}).$$

Matrices of this form occur in many image restoration contexts as a "prototype problem" and are used to model certain degradations in the recorded image.[7]

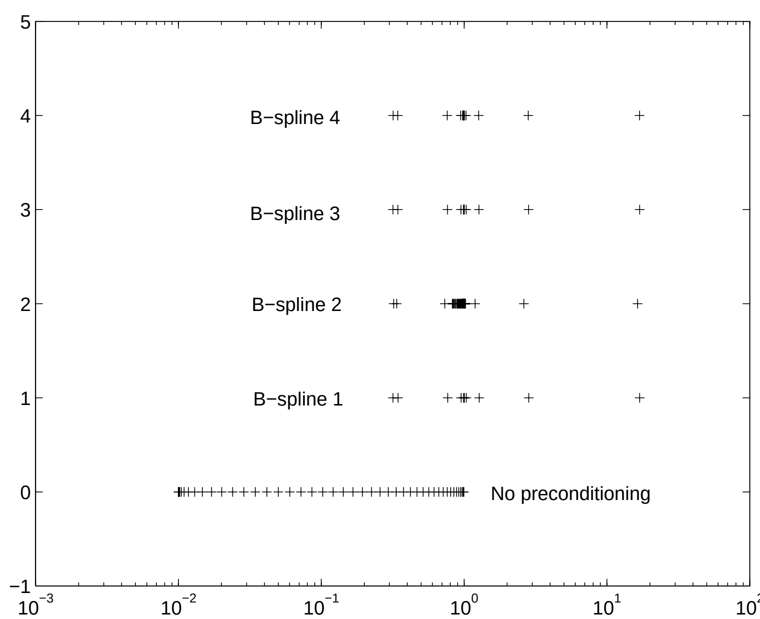

Figure 6. Singular values of different systems.

The condition number of A is approximately 2.4×10^6. Because of the ill-conditioning of A, we regularize the problem by using the identity matrix as the regularization operator. The regularization parameter is chosen to be 0.01 according to tests done by Eldén[7] who solved the same linear system by direct methods. We precondition the system by using I (no preconditioning) and B_{100}^m, with $m = 1, 2, 3$ and 4. The numbers of iterations required for convergence are 54, 8, 13, 8 and 11 respectively. In Figures 6 and 7, we show the spectra of the preconditioned systems and the convergence history.

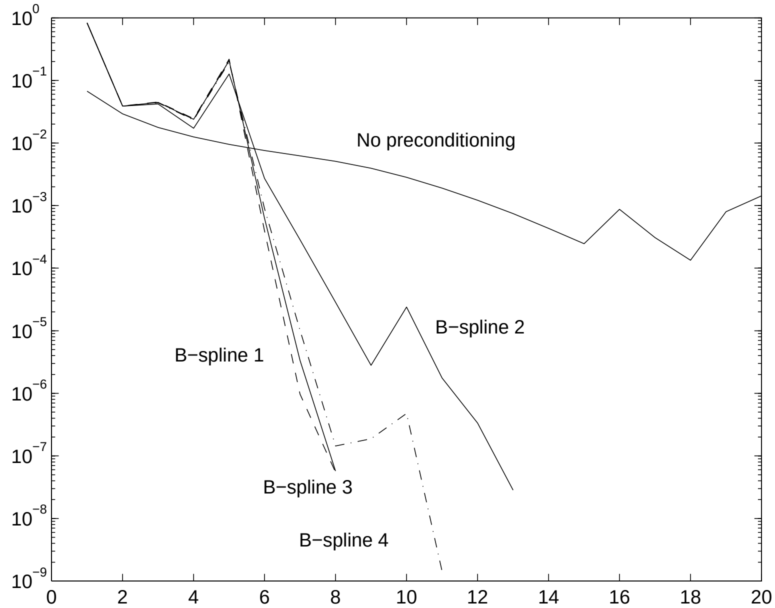

Figure 7. Convergence history for different systems.

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