Circulant Preconditioners for Ill-Conditioned Hermitian Toeplitz Matrices

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Abstract. In this paper, we propose a new family of circulant preconditioners for solving ill-conditioned Hermitian Toeplitz systems $A\mathbf{x} = \mathbf{b}$. The eigenvalues of the preconditioners are given by the convolution products of the generating function f of A with some summation kernels. When f is a nonnegative 2π-periodic continuous function defined on $[-\pi, \pi]$ with a zero of order $2p$, we show that the circulant preconditioners are positive definite and the spectrum of the preconditioned matrix is uniformly bounded in $[a, b]$ with at most $2p+2$ outliers where $0 < a < b < \infty$. Hence the linear system can be solved by the preconditioned conjugate gradient method efficiently. We emphasize that the construction of the circulant preconditioner does not require the explicit knowledge of the generating function. Numerical results are included.

1. Introduction

An n-by-n matrix $A_n = [a_{ij}]$ is said to be Toeplitz if $a_{ij} = a_{i-j}$. Toeplitz systems of the form $A_n\mathbf{x} = \mathbf{b}$ occur in a variety of applications in mathematics and engineering [**7**]. There are a number of specialized fast direct methods for solving Toeplitz systems that require $O(n^2)$ operations, see for instance [**17**]. Faster methods requiring $O(n \log^2 n)$ operations have also been developed, see [**4, 3**].

Strang in [**16**] proposed using the preconditioned conjugate gradient method with circulant matrices as preconditioners for solving Toeplitz systems. The number of operations per iteration is of order $O(n \log n)$ as circulant systems can be solved efficiently by the Fast Fourier Transform (FFT). Several successful circulant preconditioners have been introduced and analyzed; see for instance [**10, 5**]. In these papers, the Toeplitz matrix A_n is assumed to be generated by a generating function f, i.e., the diagonals of A_n are given by the Fourier coefficients of f. It has been shown that if f is a positive function in the Wiener class (i.e., the Fourier coefficients of f are absolutely summable), then these circulant preconditioned systems converge superlinearly [**5**]. However, if f has zeros, the corresponding Toeplitz system will be ill-conditioned. Tyrtyshnikov [**18**] has proved that the Strang [**16**] and the T. Chan [**10**] preconditioners will fail in this case.

1991 *Mathematics Subject Classification.* Primary 65F10, 65F15, 65T10.

Raymond Chan was supported in part by Hong Kong Research Grants Council Grant No. CUHK 4207/97 and CUHK DAG Grant No. 2060143.

Michael Ng was supported in part by Hong Kong University CRCG grant no. 10201939.

To tackle this problem, non-circulant type preconditioners have been proposed, see [**6, 8, 2, 15, 13**]. The basic idea behind these preconditioners is to find a function g that matches the zeros of f. Then the preconditioner is constructed based on the function g. These approaches work when the generating function f is given explicitly. However, in some applications, we are given only a Toeplitz system where the underlying generating function is unknown and therefore these preconditioners cannot be constructed. In contrast, circulant preconditioners only use the entries of the given Toeplitz matrix. Recently, Di Benedetto [**1**] proved that the condition numbers of the preconditioned matrices by sine transform preconditioners are uniformly bounded. However, the preconditioners themselves may be singular or indefinite in general. Our aim in this paper is to propose a family of positive definite circulant preconditioners that do not require the explicit knowledge of f and that work for ill-conditioned systems.

Chan and Yeung [**9**] showed that circulant preconditioners can be derived in general by considering convolution products of the generating function f with some kernels. For instance, the Dirichlet kernel $\mathcal{D}_{\lfloor \frac{n}{2} \rfloor}$ gives to the Strang preconditioner. They proved that for any positive 2π-periodic continuous function f, if $\mathcal{C}_n$ is a kernel such that $\mathcal{C}_n * f$ tends to f uniformly on $[-\pi, \pi]$, then the corresponding preconditioned system $C_n^{-1} A_n$ will have clustered spectrum. This result turns the problem of finding a good preconditioner to the problem of approximating f with $\mathcal{C}_n * f$. We note that $\mathcal{D}_{\lfloor \frac{n}{2} \rfloor} * f$ depends solely on the first $n-1$ Fourier coefficients $\{a_j\}_{|j|<n}$ of f. Thus the Strang preconditioner does not require explicitly knowledge of f.

Our idea here is to construct positive definite preconditioners by approximating f by $\mathcal{K}_{n,r} * f$ (defined in §2) that matches the zeros of f and depends only on $\{a_j\}_{|j|<n}$. In this proceeding paper, we mainly consider nonnegative 2π-periodic continuous function with a single zero. We will show that if $f(\theta)$ has a zero of order $2p$ at $\theta = \gamma$, then $\mathcal{K}_{n,2r} * f$ matches the zero of f for $r > p$. We will also show that the spectrum of the preconditioned matrix is uniformly bounded with at most $2p + 2$ outliers. The general case for generating functions with multiple zeros will be considered in a future paper.

The outline of the paper is as follows. In §2, we give an efficient method for computing the eigenvalues of the preconditioners. In §3 we show that $\mathcal{K}_{n,r} * f$ matches the zero of f. We then estimate in §4, the spectrum of the preconditioned matrices. Numerical results are given in §5 to illustrate the effectiveness of our preconditioners in solving ill-conditioned Toeplitz systems.

REMARK: This paper was first presented at the International Congress of Chinese Mathematicians in Beijing in December, 1998. Recently, similar results have also been obtained independently by Potts and Steidl [**14**].

2. Construction of Circulant Preconditioners

Denote the space of all 2π-periodic continuous real-valued functions by $\mathbf{C}_{2\pi}$. The Fourier coefficients of a function f in $\mathbf{C}_{2\pi}$ are given by

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \, d\theta, \quad k = 0, \pm 1, \pm 2, \cdots .$$

The n-by-n Hermitian Toeplitz matrix generated by f will be denoted by $A_n[f] = [a_{i-j}]$. We remark that $A_n[f]$ is positive definite when f is a nonnegative function that takes positive values on a nondegenerate interval, see [**6**]. Suppose θ_0 is a zero of f. We say that θ_0 is of *order* p if p is the smallest positive integer such that $f^{(p+1)}(\theta)$ is continuous in a neighborhood of θ_0 and $f^{(p)}(\theta_0) \neq 0$. We note that when f is nonnegative, $f^{(p)}(\theta_0) > 0$ and p must be even.

For the Toeplitz systems considered in this paper, we will assume that f is a nonnegative function with a single zero. The systems will be solved by using preconditioned conjugate gradient (PCG) methods with circulant preconditioners. It is well known that circulant matrices can be diagonalized by the Fourier matrix F_n, see [**7**]. Therefore, a circulant matrix is uniquely determined by its set of eigenvalues. For a given function f, we define the circulant preconditioner $C_n[f]$ to be an n-by-n matrix with its j-th eigenvalue given by

$$\lambda_j(C_n[f]) = f(\frac{2\pi j}{n}), \quad 0 \leq j < n. \tag{2.1}$$

We note that $C_n[f] = F_n^* \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) F_n$, see [**7**].

In the following, we will use the kernel function

$$\mathcal{K}_{n,2r}(\theta) = \frac{k_{n,2r}}{n^{r-1}} \left(\frac{\sin(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})} \right)^{2r}, \quad r = 1, 2, \ldots \tag{2.2}$$

to construct our circulant preconditioners. Here $k_{n,2r} (= O(1))$ is a normalization constant such that the integral of $\mathcal{K}_{n,2r}$ over $[-\pi, \pi]$ is 1. We note that $\mathcal{K}_{n,2}$ is the Fejér kernel and $\mathcal{K}_{n,4}$ is the Jackson kernel [**12**, p.57].

For any N, the Fejér kernel $\mathcal{K}_{N,2}$ can be expressed as

$$\mathcal{K}_{N,2}(\theta) = \sum_{k=-N+1}^{N-1} b_k^{(N,2)} e^{ik\theta},$$

where

$$b_k^{(N,2)} = \frac{N - |k|}{2\pi N}, \quad k = 0, \pm 1, \pm 2, \cdots, \pm N-1.$$

Hence by (2.2),

$$\mathcal{K}_{N,2r}(\theta) = \sum_{k=-r(N-1)}^{r(N-1)} b_k^{(N,2r)} e^{ik\theta} \tag{2.3}$$

where the coefficients $b_k^{(N,2r)}$ can be obtained by convolving the vector $(b_k^{(N,2)})_{k=0}^{N-1}$ with itself for $r-1$ times using FFTs. Hence the cost of constructing the coefficients is of $O(rN \log N)$ operations.

The convolution product of $\mathcal{K}_{n,2r}$ with a function f is defined as

$$(\mathcal{K}_{n,2r} * f)(\theta) \equiv \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) f(t - \theta) \, dt. \tag{2.4}$$

Since $f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, by (2.3), the convolution product in (2.4) becomes

$$(\mathcal{K}_{N,2r} * f)(\theta) = \sum_{k=-r(N-1)}^{r(N-1)} a_k b_k e^{ik\theta}.$$

Hence our circulant preconditioners $C_n[\mathcal{K}_{N,2r} * f]$ can be constructed by using only the first $r(N-1)+1$ coefficients of f. Thus when we are given a Toeplitz system $A_n \mathbf{x} = \mathbf{b}$, we can choose the largest N such that $r(N-1) < n$. Then $\mathcal{K}_{N,2r} * f$

only requires a_k for $|k| < n$. This means that $C_n[\mathcal{K}_{N,2r} * f]$ can be constructed efficiently without the explicit knowledge of the underlying generating function f.

3. Properties of the Kernel $\mathcal{K}_{n,2r}$

In this section, we study some properties of $\mathcal{K}_{n,2r}$ in order to see how good the approximation of f by $\mathcal{K}_{n,2r} * f$ will be. We first note the following Lemma.

LEMMA 3.1. [**12**, p.55] *Let p and r be positive integers with $r > p$. Then*

$$\left(\mathcal{K}_{n,2r} * \theta^{2p}\right)(0) = O(\frac{1}{n^{2p}}).$$

Next we simplify $(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p})(\phi)$ where $\theta_{2\pi}$ is the periodic extension of θ on $[-\pi, \pi]$.

LEMMA 3.2. *Let p be a positive integer. For any $\phi \in [-\pi, -\pi/2]$, there exist positive α_1 and β_1 such that*

$$\alpha_1 \leq \frac{\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\left[\mathcal{K}_{n,2r} * \theta^{2p}(\theta + 2\pi)^{2p}\right](\phi)} \leq \beta_1, \quad \forall t \in [-\pi, \pi].$$

For any $\phi \in [-\pi/2, \pi/2]$, there exist positive α_2 and β_2 such that

$$\alpha_2 \leq \frac{\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\left(\mathcal{K}_{n,2r} * \theta^{2p}\right)(\phi)} \leq \beta_2, \quad \forall t \in [-\pi, \pi].$$

For any $\phi \in [\pi/2, \pi]$, there exist positive α_3 and β_3 such that

$$\alpha_3 \leq \frac{\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\left(\mathcal{K}_{n,2r} * \theta^{2p}(\theta - 2\pi)^{2p}\right)(\phi)} \leq \beta_3, \quad \forall t \in [-\pi, \pi]. \tag{3.1}$$

PROOF. We just note that for any $\phi \in [-\pi, -\pi/2]$, there exist positive α_1 and β_1 such that

$$\alpha_1 \leq \frac{(\phi - t)_{2\pi}^{2p}}{(\phi - t)^{2p}(\phi + 2\pi - t)^{2p}} \leq \beta_1, \quad \forall t \in [-\pi, \pi].$$

For any $\phi \in [-\pi/2, \pi/2]$, there exist positive α_2 and β_2 such that

$$\alpha_2 \leq \frac{(\phi - t)_{2\pi}^{2p}}{(\phi - t)^{2p}} \leq \beta_2, \quad \forall t \in [-\pi, \pi]. \tag{3.2}$$

And for any $\phi \in [\pi/2, \pi]$, there exist positive α_3 and β_3 such that

$$\alpha_3 \leq \frac{(\phi - t)_{2\pi}^{2p}}{(\phi - t)^{2p}(\phi - 2\pi + t)^{2p}} \leq \beta_3, \quad \forall t \in [-\pi, \pi].$$

$\square$

With Lemmas 3.1 and 3.2, we have the following theorem.

THEOREM 3.3. *Let p and r be positive integers with $r > p$. Then there exist positive numbers α and β and a natural number M such that*

$$\alpha \leq \frac{\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} \leq \beta \tag{3.3}$$

for all $n > M$ and $\pi/n < |\phi| \leq \pi$.

PROOF. By Lemma 3.2, for different values of ϕ, $(\mathcal{K}_{n,2r}*\theta_{2\pi}^{2p})(\phi)$ can be replaced by different functions. Hence, we proceed the proof for different ranges of values of ϕ.

We note from (3.2) that $\phi_{2\pi}^{2p} = \phi^{2p} g(\phi)$ for some bounded positive function g. For $\phi \in [\pi/n, \pi/2]$, $(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p})(\phi) = [\mathcal{K}_{n,2r} * \theta^{2p} g(\theta)](\phi)$. By the Mean Value Theorem, there exists $\zeta \in [-\pi, \pi]$ such that

$$[\mathcal{K}_{n,2r} * \theta^{2p} g(\theta)](\phi) = g(\zeta)(\mathcal{K}_{n,2r} * \theta^{2p})(\phi).$$

It follows that

$$\begin{aligned} \frac{\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} &= \frac{g(\zeta)\left(\mathcal{K}_{n,2r} * \theta^{2p}\right)(\phi)}{g(\phi)\phi^{2p}} \\ &= \frac{g(\zeta)}{g(\phi)} \sum_{k=0}^{p} \binom{2p}{2k} \phi^{-2k} \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) t^{2k}\, dt. \end{aligned}$$

By applying Lemma 3.1, $\int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t)t^{2k}\, dt = c_k/n^{2k}$, for $k = 0, \cdots, p$ where $c_k = O(1) > 0$. Then

$$\frac{\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} = \frac{g(\zeta)}{g(\phi)} \sum_{k=0}^{p} \frac{d_k}{\phi^{2k} n^{2k}},$$

where $d_k = c_k \binom{2p}{2k}$. Let $g_{\max} = \max_{\phi\in[\pi/n,\pi/2]} g(\phi)$ and $g_{\min} = \min_{\phi\in[\pi/n,\pi/2]} g(\phi)$. Since

$$0 < 1 = d_0 \le \sum_{k=0}^{p} \frac{d_k}{\phi^{2k} n^{2k}} \le \sum_{k=0}^{p} \frac{d_k}{\pi^{2k}},$$

we have

$$\frac{g(\zeta)}{g_{\max}} \le \frac{\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} \le \frac{g(\zeta)}{g_{\min}} \sum_{k=0}^{p} \frac{d_k}{\pi^{2k}}.$$

For $\phi \in [\pi/2, \pi]$, we note that $(\pi/2)^{2p} \le \phi_{2\pi}^{2p} \le \pi^{2p}$. By (3.1),

$$\left(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p}\right)(\phi) = h(\phi)\left[\mathcal{K}_{n,2r} * \theta^{2p}(\theta - 2\pi)^{2p}\right](\phi),$$

for some positive bounded function h. On the other hand,

$$\begin{aligned} \left[\mathcal{K}_{n,2r} * \theta^{2p}(\theta - 2\pi)^{2p}\right](\phi) &= \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t)(\phi - t)^{2p}(\phi - t - 2\pi)^{2p}\, dt \\ &= \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t)\left(\phi^{2p}(\phi - 2\pi)^{2p} + s(t)\right)\, dt, \end{aligned}$$

where $s(t)$ is a degree $4p$ polynomial such that $s(0) = s'(0) = 0$. By Lemma 3.1, $\int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t)s(t)\, dt = O(1/n^{2j})$ for some $j \ge 1$. Thus,

$$\frac{\left[\mathcal{K}_{n,2r} * \theta^{2p}(\theta - 2\pi)^{2p}\right](\phi)}{\phi_{2\pi}^{2p}} = (\phi - 2\pi)^{2p} + O(\frac{1}{n^2}).$$

It follows that for sufficiently large n, the above ratio is bounded below and above by some positive constants.

For the case $\phi \in [-\pi, -\pi/2]$, it can be proved in a similar way as above that for sufficiently large n, (3.3) holds. The last case $\phi \in [-\pi/2, -\pi/n]$ is similar to the case where $\phi \in [\pi/n, \pi/2]$. $\square$

Using the fact that $[\mathcal{K}_{n,2r} * (\theta-\gamma)^{2p}_{2\pi}](\phi) = (\mathcal{K}_{n,2r} * \theta^{2p}_{2\pi})(\phi-\gamma)$, we obtain the following corollary which deals with functions having a zero at $\gamma \neq 0$.

COROLLARY 3.4. *Let $\gamma \in [-\pi,\pi]$, p and r be positive integers with $r > p$. Then there exist positive numbers α and β and a natural number M such that*

$$\alpha \le \frac{\left[\mathcal{K}_{n,2r} * (\theta-\gamma)^{2p}_{2\pi}\right](\phi)}{(\phi-\gamma)^{2p}_{2\pi}} \le \beta$$

for all $n > M$ and $\pi/n < |\phi-\gamma| \le \pi$.

Now we are in the position of proving a main result for functions with a single zero of order $2p$.

THEOREM 3.5. *Let $f \in \mathbf{C}_{2\pi}$ be a nonnegative function with a zero of order $2p$ at $\gamma \in [-\pi,\pi]$ and let $r > p$ be any integer. Then there exist positive numbers α and β and a natural number M such that*

$$\alpha \le \frac{(\mathcal{K}_{n,2r} * f)(\phi)}{f(\phi)} \le \beta$$

for all $n > M$ and $\pi/n < |\phi-\gamma| \le \pi$.

PROOF. The result can be obtained by writing

$$\frac{(\mathcal{K}_{n,2r} * f)(\phi)}{f} = \frac{(\mathcal{K}_{n,2r} * f)(\phi)}{\left[\mathcal{K}_{n,2r} * (\theta-\gamma)^{2p}_{2\pi}\right](\phi)} \cdot \frac{\left[\mathcal{K}_{n,2r} * (\theta-\gamma)^{2p}_{2\pi}\right](\phi)}{(\phi-\gamma)^{2p}_{2\pi}} \cdot \frac{(\phi-\gamma)^{2p}_{2\pi}}{f},$$

and then applying the Mean Value Theorem to the first factor, Corollary 3.4 to the second factor, and the definition of the zero (cf. §2) to the third factor. □

So far we have considered only the interval $\pi/n < |\phi-\gamma| \le \pi$. For $|\phi-\gamma| \le \pi/n$, we have the following theorem.

THEOREM 3.6. *Let $f \in \mathbf{C}_{2\pi}$ be a nonnegative function with a zero of order $2p$ at $\gamma \in [-\pi,\pi]$ and let $r > p$ be any integer. Then for any $|\phi-\gamma| \le \pi/n$, we have*

$$(\mathcal{K}_{n,2r} * f)(\phi) = O\left(\frac{1}{n^{2p}}\right).$$

PROOF. We first prove the theorem for the function $f(\phi) = \phi^{2p}_{2\pi}$. In view of (3.2), there exists a positive continuous function $g(t)$ such that $(\phi-t)^{2p}_{2\pi} = g(t)(\phi-t)^{2p}$. Hence,

$$\begin{aligned}
\left(\mathcal{K}_{n,2r} * \theta^{2p}_{2\pi}\right)(\phi) &= \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) g(t)(\phi-t)^{2p}\, dt \\
&= g(\zeta) \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t)(\phi-t)^{2p}\, dt \\
&= g(\zeta) \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) \sum_{j=0}^{2p} \binom{2p}{j} \phi^{2p-j}(-t)^j\, dt,
\end{aligned}$$

where the second equality is obtained from the Mean Value Theorem for some $\zeta \in [-\pi, \pi]$. For odd j, $\int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t)t^j \, dt = 0$. Thus

$$
\begin{aligned}
\left(\mathcal{K}_{n,2r} * \theta^{2p}\right)(\phi) &= g(\zeta) \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) \sum_{j=0}^{p} \binom{2p}{2j} \phi^{2p-2j} t^{2j} \, dt \\
&\le g(\zeta) \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) \sum_{j=0}^{p} \binom{2p}{2j} \left(\frac{\pi}{n}\right)^{2p-2j} t^{2j} \, dt \\
&= g(\zeta) \sum_{j=0}^{p} \binom{2p}{2j} \left(\frac{\pi}{n}\right)^{2p-2j} \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) t^{2j} \, dt.
\end{aligned}
\tag{3.4}
$$

By Lemma 3.1, $\int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t)t^{2j} \, dt = O(1/n^{2j})$ for each j. Hence $\left(\mathcal{K}_{n,2r} * \theta^{2p}\right)(\phi) \le O(1/n^{2p})$. On the other hand, from (3.4),

$$
(\mathcal{K}_{n,2r} * \theta^{2p})(\phi) \ge g(\zeta) \int_{-\pi}^{\pi} \mathcal{K}_{n,2r}(t) t^{2p} \, dt = g(\zeta) O(\frac{1}{n^{2p}}).
$$

Thus the theorem holds for $f(\phi) = \phi_{2\pi}^{2p}$.

If $f(\phi) = (\phi - \gamma)_{2\pi}^{2p} g(\phi)$ for some positive function $g \in \mathbf{C}_{2\pi}$, then by the Mean Value Theorem, there exists a $\zeta \in [-\pi, \pi]$ such that

$$
\begin{aligned}
(\mathcal{K}_{n,2r} * f)(\phi) &= [\mathcal{K}_{n,2r} * (\theta - \gamma)_{2\pi}^{2p} g(\theta)](\phi) \\
&= g(\zeta)[\mathcal{K}_{n,2r} * (\theta - \gamma)_{2\pi}^{2p}](\phi) = g(\zeta)(\mathcal{K}_{n,2r} * \theta_{2\pi}^{2p})(\phi - \gamma).
\end{aligned}
$$

Hence the theorem follows. □

4. Spectral Properties of the Preconditioned Matrices

In this section, we analyze the spectra of the preconditioned matrices. We will need the following lemma.

LEMMA 4.1. [**13**, Lemma 3.2] *Let $f, g \in \mathbf{C}_{2\pi}$ be nonnegative such that $0 < a \le f/g \le b$. Then*

$$
a \le \frac{\mathbf{x}^* A_n[f] \mathbf{x}}{\mathbf{x}^* A_n[g] \mathbf{x}} \le b, \qquad \forall \mathbf{x} \ne \mathbf{0}.
$$

Finally, we have our main theorem.

THEOREM 4.2. *Let $f \in \mathbf{C}_{2\pi}$ be a nonnegative function with a zero of order $2p$ at γ. Then for all $r > p$, there exists an integer M such that for $n > M$, at most $2p + 2$ eigenvalues of $C_n^{-1}[\mathcal{K}_{n,2r} * f] A_n[f]$ are not contained in the interval $[a, b]$ where $0 < a < b < \infty$.*

PROOF. Throughout this proof, we denote by $R_n(m)$ an arbitrary n-by-n matrix of rank at most m. For any function g, we let $\tilde{C}_n[g]$ to be the circulant matrix whose j-th eigenvalue is given by

$$
\lambda_j(\tilde{C}_n[g]) = \begin{cases} \dfrac{1}{n^{2p}}, & \text{if } \left| \dfrac{2\pi j}{n} - \gamma \right| \le \dfrac{\pi}{n} \\ g(\dfrac{2\pi j}{n}), & \text{otherwise.} \end{cases}
\tag{4.1}
$$

Note that $\tilde{C}_n[g] = C_n[g] + F_n^* \Phi_n F_n$ where $C_n[g]$ is defined in (2.1) and Φ_n is a diagonal matrix of rank not more than 2.

We use the following decomposition to prove the theorem:

$$\begin{aligned}
&\frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{n,2r} * f]\mathbf{x}} \\
&= \frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* A_n\left[\sin^{2p}\left(\frac{\phi-\gamma}{2}\right)\right]\mathbf{x}} \cdot \frac{\mathbf{x}^* A_n\left[\sin^{2p}\left(\frac{\phi-\gamma}{2}\right)\right]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n\left[\sin^{2p}\left(\frac{\phi-\gamma}{2}\right)\right]\mathbf{x}} \cdot \\
&\quad \frac{\mathbf{x}^* \tilde{C}_n\left[\sin^{2p}\left(\frac{\phi-\gamma}{2}\right)\right]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n[f]\mathbf{x}} \cdot \frac{\mathbf{x}^* \tilde{C}_n[f]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n[\mathcal{K}_{n,2r} * f]\mathbf{x}} \cdot \frac{\mathbf{x}^* \tilde{C}_n[\mathcal{K}_{n,2r} * f]\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{n,2r} * f]\mathbf{x}}.
\end{aligned} \tag{4.2}$$

By Lemma 4.1, the first factor in the right hand side of (4.2) is uniformly bounded. For the second factor, since $C_n[\sin^{2p}\theta]$ is just the Strang preconditioner for $A_n[\sin^{2p}\theta]$ when $n > 2p$, we have

$$\begin{aligned}
A_n\left[\sin^{2p}\left(\frac{\phi-\gamma}{2}\right)\right] &= C_n\left[\sin^{2p}\left(\frac{\phi-\gamma}{2}\right)\right] + R_n(2p) \\
&= \tilde{C}_n\left[\sin^{2p}\left(\frac{\phi-\gamma}{2}\right)\right] + R_n(2p+2).
\end{aligned}$$

By the definition of zeros, the third factor in (4.2) is uniformly bounded. By Theorem 3.5 and the definition of $\tilde{C}_n$ in (4.1), the fourth factor in (4.2) is uniformly bounded. Finally, by Theorem 3.6 and (4.1) again, we see that the last factor of (4.2) is also uniformly bounded.

The theorem now follows by using the properties of Rayleigh quotient and the Weyl's theorem [**11**, p.184]. □

We emphasize that we do not need the explicit knowledge of where the zero γ is in order to construct the preconditioners $C_n[\mathcal{K}_{n,2r} * f]$. Moreover, since $\mathcal{K}_{n,2r}$ are positive kernels, the preconditioners $C_n[\mathcal{K}_{n,2r} * f]$ are positive definite, see [**9**].

By Theorem 4.2, the number of PCG iterations required for convergence is of $O(\log n)$, see [**1**]. Since each PCG iteration and also the construction of the preconditioners both require $O(n \log n)$ operations, the total complexity of our PCG method is at most $O(n \log^2 n)$.

5. Numerical Experiments

In this section, we illustrate by numerical examples the effectiveness of the preconditioners $C_n[\mathcal{K}_{n,2r} * f]$ in solving ill-conditioned Toeplitz systems. For comparisons, we also test the Strang and the T. Chan circulant preconditioners, see [**16, 10**]. We solve Toeplitz systems $A_n[f]\mathbf{x} = \mathbf{b}$ by the preconditioned conjugate gradient method for eight test functions: θ^2, $(\theta^2-1)^2$, $\theta^2(\pi^2-\theta^2)$, $\theta^2(\pi^4-\theta^4)$, θ^4, $\theta^4(\pi^2-\theta^2)$,

$$\sum_{|k|\le 1024} \frac{1}{|k|+1} e^{ik\theta} - 0.3853, \quad \text{and} \quad \sum_{|k|\le 1024} \frac{1}{|k|^{0.5}+1} e^{ik\theta} - 0.4134.$$

Regarding the last two functions, we found numerically that the minimum values of $\sum_{|k|\le 1024} \frac{1}{|k|+1} e^{ik\theta}$ and $\sum_{|k|\le 1024} \frac{1}{|k|^{0.5}+1} e^{ik\theta}$ are approximately equal to 0.3853 and 0.4134 respectively. Hence the two test functions are approximately zero at some points in $[-\pi, \pi]$.

	θ^2							$(\theta^2-1)^2$						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	16	36	79	170	362	753	†	21	53	141	293	547	†	†
S	6	7	7	7	7	7	8	11	10	12	9	10	8	12
T	11	12	16	19	23	29	39	15	18	24	30	27	36	46
$K_{N,4}$	9	8	9	10	9	9	9	13	13	13	14	12	13	11
$K_{N,6}$	9	10	10	10	10	9	9	13	13	13	14	14	13	13
$K_{N,8}$	10	9	10	10	10	10	10	14	13	13	15	15	14	13

	$\theta^2(\pi^2-\theta^2)$							$\theta^2(\pi^4-\theta^4)$						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	16	32	61	116	220	428	835	16	32	64	128	256	510	†
S	9	9	9	9	10	11	12	8	9	9	9	10	11	12
T	11	12	14	17	20	26	33	11	12	15	17	22	27	38
$K_{N,4}$	10	10	11	11	11	11	11	10	10	11	11	11	11	11
$K_{N,6}$	10	10	11	11	11	11	12	10	10	11	11	11	11	13
$K_{N,8}$	11	11	12	12	11	12	13	11	11	12	12	12	12	12

	θ^4							$\theta^4(\pi^2-\theta^2)$						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	22	63	209	790	†	†	†	16	46	131	410	†	†	†
S	12	13	18	21	24	31	41	12	13	16	20	22	31	38
T	17	26	42	71	161	167	247	16	24	35	58	106	144	196
$K_{N,4}$	13	15	17	20	24	26	26	14	15	16	20	22	27	26
$K_{N,6}$	13	15	16	18	18	17	18	14	15	16	18	18	18	21
$K_{N,8}$	14	16	17	19	19	19	20	16	16	18	19	20	21	23

	$\sum_{\lvert k\rvert<1024} 1/(\lvert k\rvert+1)e^{ik\theta}-0.3853$							$\sum_{\lvert k\rvert<1024} 1/(\lvert k\rvert^{0.5}+1)e^{ik\theta}-0.4134$						
n	16	32	64	128	256	512	1024	16	32	64	128	256	512	1024
I	19	45	112	184	240	296	343	18	41	92	238	715	†	†
S	10	10	10	7	8	8	8	9	11	10	9	10	9	8
T	12	15	15	14	12	10	8	11	14	13	12	17	15	13
$K_{N,4}$	7	8	8	7	8	8	7	10	11	10	9	9	9	8
$K_{N,6}$	7	10	7	8	8	8	8	10	11	10	9	10	9	9
$K_{N,8}$	9	9	8	8	8	8	8	11	11	11	10	10	10	9

TABLE 1. Numbers of iterations for different preconditioners.

In our test, we use $\mathcal{K}_{N,2r}$ as kernel functions where N is the largest integer such that $N(r-1) < n$. The right-hand side vector $\mathbf{b}$ is formed by multiplying a random vector to $A_n[f]$. The initial guess is the zero vector and the stopping criteria is $||\mathbf{r}_q||_2/||\mathbf{r}_0||_2 \le 10^{-7}$ where $\mathbf{r}_q$ is the residual vector after q iterations. All computations are done by Matlab on a Sun Ultra SPARC 1 workstation.

Table 1 shows the numbers of iterations required for convergence for different choices of preconditioners. In the table, I denotes no preconditioner, S is the Strang preconditioner, T is the T. Chan preconditioner, and $K_{N,2r}$ are the preconditioners from $\mathcal{K}_{N,2r}$ with $r = 2$, 3 and 4. We recall that T is just $K_{N,2}$. For those iterations more than 1000, we denote them by †.

The first four test functions have single or multiple zeros of order 2 on $[-\pi, \pi]$. We see that T. Chan preconditioner does not work. This can be explained by the fact that the order of $\mathcal{K}_{N,2} * f$ does not match that of f at the zeros of f. The Strang and T. Chan preconditioners do not work for the next two test functions which are functions with a zero of order 4. As expected, $K_{N,6}$ and $K_{N,8}$ still work very well. The last two functions are functions with slowly decaying Fourier coefficients. It is nontrivial to determine the order of their zeros. However, the results show that $K_{N,r}$ perform better than the Strang and T. Chan preconditioners.

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