CIRCULANT PRECONDITIONERS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider the solution of ordinary differential equations using boundary value methods. These methods require the solutions of one or more unsymmetric, large and sparse linear systems. Krylov subspace methods with the Strang block-circulant preconditioners are proposed for solving these linear systems. We prove that our preconditioners are invertible and all the eigenvalues of the preconditioned systems are clustered around 1. Therefore we expect fast convergence when Krylov subspace methods such as the GMRES method are applied to solving these preconditioned systems. Numerical results are reported to illustrate the effectiveness of our methods.

Key words. Circulant preconditioner, linear multistep formula, ordinary differential equation, initial value problem, boundary value method

AMS subject classifications. 65F10, 65N22, 65L05, 65F15, 15A18

1. Introduction.

In this paper, we consider the solution of ordinary differential equations (ODEs) by using boundary value methods (BVMs), see [2]. BVMs are a class of numerical methods based on the linear multistep formulae (LMF). The solution to a given ODE is obtained by solving a linear system

$$M\mathbf{x} = \mathbf{b}. \tag{1}$$

Recently, Bertaccini [1] has proposed to use circulant preconditioners in conjunction with Krylov subspace methods to solve (1).

Circulant matrices, as preconditioners for Toeplitz systems in the preconditioned conjugate gradient method, have been studied extensively since 1986, see [3]. We remark that a matrix T is said to be Toeplitz if its entries are constant along its diagonals and a matrix C is called circulant if it is Toeplitz and the last entry of every row is the first entry of its succeeding row. It has been shown that the circulant preconditioners are good preconditioners for the solutions of a large class of Toeplitz systems, see [3].

In this paper, we consider the linear initial value problem

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = J_m\mathbf{y}(t) + \mathbf{g}(t), \quad t \in (t_0, T], \\ \\ \mathbf{y}(t_0) = \mathbf{v}, \end{cases} \tag{2}$$

where $\mathbf{y}(t)$, $\mathbf{g}(t) : \mathbb{R} \to \mathbb{R}^m$, $\mathbf{v} \in \mathbb{R}^m$, and $J_m \in \mathbb{R}^{m\times m}$. Using BVMs on (2), the matrix M in (1) can be written as

$$M = A \otimes I_m - hB \otimes J_m,$$

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where A and B are $(s+1)$-by-$(s+1)$ matrices from the LMF, I_m is the m-by-m identity matrix and $h = (T - t_0)/s$ is the integration step size. We note that both A and B are sums of a Toeplitz matrix plus a low rank matrix. The size of M can be very large when h is small and/or m is large. If a direct method is used to solve the system, the operation count will be too expensive and slow for practical, large scale applications.

In [1], Bertaccini proposed to use iterative methods such as the GMRES method [5] to solve (1). In order to speed up the convergence rate of the method, he tried T. Chan block-circulant preconditioner [3] and a circulant matrix, which he called the P-circulant preconditioner, to precondition the matrix M. His numerical results showed that the preconditioned systems from both circulant preconditioners converge very quickly. However, he can only show that the P-circulant preconditioner is invertible.

In this paper, we propose a preconditioner of the form

$$S = s(A) \otimes I_m - hs(B) \otimes J_m,$$

where $s(A)$ and $s(B)$ are the Strang preconditioners for A and B respectively, see [3]. We show that if an A_{k_1,k_2}-stable BVM is used to discretize (2) and that all the eigenvalues of J_m are in the negative half of the complex plane, then S is invertible and the preconditioned matrix can be written as

$$S^{-1}M = I_{m(s+1)} + L,$$

where the rank of L is at most $2m(k_1 + k_2)$. We therefore expect fast convergence if Krylov subspace methods are used to solve the preconditioned system, see [4, p.54]. We remark that our proposed preconditioner preserves important algebraic properties in the matrix M, such as the invertibility and the tensor and sparsity pattern.

In contrast, Bertaccini [1] showed that, with his P-circulant preconditioner P,

$$P^{-1}M = I_{m(s+1)} + U + V,$$

where rank $U \leq 2m(k_1 + k_2)$ and the 2-norm of V is of order $O(1)$. Because of this extra term V, we expect the performance of our preconditioner to be better than that of the P-circulant preconditioner. Numerical results in §5 will illustrate the effectiveness of our preconditioner.

The paper is organized as follows. In §2, we recall the general scheme of BVMs. We introduce our circulant preconditioner in §3 and give the convergence analysis of our method in §4. Numerical examples are given in §5. Finally, §6 gives some concluding remarks.

2. BVMs and their Matrix Forms. BVMs are methods based on LMF for solving ODEs, see [2]. Given (2), a BVM approximates its solution by means of a discrete boundary value problem. By using a μ-step LMF over a uniform mesh

$$t_j = t_0 + jh, \quad j = 0, \cdots, s,$$

with $h = (T - t_0)/s$, we have

$$\sum_{i=-\nu}^{\mu-\nu} \alpha_{i+\nu} \mathbf{y}_{n+i} = h \sum_{i=-\nu}^{\mu-\nu} \beta_{i+\nu} \mathbf{f}_{n+i}, \quad n = \nu, \ldots, s - \mu + \nu. \tag{3}$$

Here, $\mathbf{y}_n$ is the discrete approximation to $\mathbf{y}(t_n)$, $\mathbf{f}_n = J_m \mathbf{y}_n + \mathbf{g}_n$ and $\mathbf{g}_n = \mathbf{g}(t_n)$.

The BVM in (3) must be used with ν initial conditions and $\mu-\nu$ final conditions. That is, we need the values $\mathbf{y}_0, \cdots, \mathbf{y}_{\nu-1}$ at $t=t_0$ and the values $\mathbf{y}_{n+\mu-\nu-1}, \cdots, \mathbf{y}_n$ at $t=T$. The initial condition in (2) only gives us one value. In order to obtain the other initial and final values, we have to provide additional $(\mu-1)$ equations:

$$\sum_{i=0}^{\mu} \alpha_i^{(j)} \mathbf{y}_i = h \sum_{i=0}^{\mu} \beta_i^{(j)} \mathbf{f}_i, \quad j=1,\ldots,\nu-1, \tag{4}$$

and

$$\sum_{i=0}^{\mu} \alpha_{k-i}^{(j)} \mathbf{y}_{s-i} = h \sum_{i=0}^{\mu} \beta_{k-i}^{(j)} \mathbf{f}_{s-i}, \quad j=s-\mu+\nu+1,\ldots,s. \tag{5}$$

The coefficients $\alpha^{(j)}$ and $\beta^{(j)}$ in (4) and (5) should be chosen such that the truncation errors of (4) and (5) are of the same order as that in (3), see [2, p,132]. By combining (3), (4) and (5), we obtain a linear system as in (1). The advantage in using BVMs is that they have much better stability properties than traditional initial value methods, see [2, p.79].

In matrix terms, the discrete problem (1) generated by (3)–(5) is given by

$$M\mathbf{y} \equiv (A \otimes I_m - hB \otimes J_m)\mathbf{y} = \mathbf{e} \otimes \mathbf{v} + h(B \otimes J_m)\mathbf{g}, \tag{6}$$

where

$$\begin{aligned} \mathbf{e} &= (1, 0, \cdots, 0)^t \in \mathbb{R}^{(s+1)}, \\ \mathbf{y} &= (\mathbf{y}_0, \cdots, \mathbf{y}_s)^t \in \mathbb{R}^{(s+1)m}, \\ \mathbf{g} &= (\mathbf{g}_0, \cdots, \mathbf{g}_s)^t \in \mathbb{R}^{(s+1)m}, \end{aligned}$$

and A and B are $(s+1)$-by-$(s+1)$ matrices given by:

$$A = \begin{pmatrix}
1 & \cdots & 0 & & & & \\
\alpha_0^{(1)} & \cdots & \alpha_\mu^{(1)} & & & & \\
\vdots & \vdots & \vdots & & & 0 & \\
\alpha_0^{(\nu-1)} & \cdots & \alpha_\mu^{(\nu-1)} & & & & \\
\alpha_0 & \cdots & \alpha_\mu & & & & \\
 & \alpha_0 & \cdots & \alpha_\mu & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \\
 & & & & \alpha_0 & \cdots & \alpha_\mu \\
 & & 0 & & \alpha_0^{(s-\mu+\nu+1)} & \cdots & \alpha_\mu^{(s-\mu+\nu+1)} \\
 & & & & \vdots & \vdots & \vdots \\
 & & & & \alpha_0^{(s)} & \cdots & \alpha_\mu^{(s)}
\end{pmatrix}$$

and

$$
B = \left(\begin{array}{ccccccc}
0 & \cdots & 0 & & & & \\
\beta_0^{(1)} & \cdots & \beta_\mu^{(1)} & & & & \\
\vdots & \vdots & \vdots & & & & \\
\beta_0^{(\nu-1)} & \cdots & \beta_\mu^{(\nu-1)} & & & 0 & \\
\beta_0 & \cdots & \beta_\mu & & & & \\
 & \beta_0 & \cdots & \beta_\mu & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \\
 & & & & \beta_0 & \cdots & \beta_\mu \\
 & & 0 & & \beta_0^{(s-\mu+\nu+1)} & \cdots & \beta_\mu^{(s-\mu+\nu+1)} \\
 & & & & \vdots & \vdots & \vdots \\
 & & & & \beta_0^{(s)} & \cdots & \beta_\mu^{(s)}
\end{array}\right).
$$

3. Construction of the Preconditioner. We propose the following preconditioner for the matrix M in (6):

$$
S = s(A) \otimes I_m - hs(B) \otimes J_m, \tag{7}
$$

where $s(A)$ and $s(B)$ are given by

$$
s(A) = \left(\begin{array}{ccccccccc}
\alpha_\nu & \cdots & \alpha_\mu & & & & \alpha_0 & \cdots & \alpha_{\nu-1} \\
\vdots & \ddots & & \ddots & & & & \ddots & \vdots \\
\alpha_0 & & \ddots & & \ddots & & & & \alpha_0 \\
 & \ddots & & \ddots & & \ddots & & 0 & \\
 & & \ddots & & \ddots & & \ddots & & \\
 & 0 & & \ddots & & \ddots & & \ddots & \\
\alpha_\mu & & & & \ddots & & \ddots & & \alpha_\mu \\
\vdots & \ddots & & & & \ddots & & \ddots & \vdots \\
\alpha_{\nu+1} & \cdots & \alpha_\mu & & & & \alpha_0 & \cdots & \alpha_\nu
\end{array}\right),
$$

and

$$
s(A) = \left(\begin{array}{ccccccccc}
\beta_\nu & \cdots & \beta_\mu & & & & \beta_0 & \cdots & \beta_{\nu-1} \\
\vdots & \ddots & & \ddots & & & & \ddots & \vdots \\
\beta_0 & & \ddots & & \ddots & & & & \beta_0 \\
 & \ddots & & \ddots & & \ddots & & 0 & \\
 & & \ddots & & \ddots & & \ddots & & \\
 & 0 & & \ddots & & \ddots & & \ddots & \\
\beta_\mu & & & & \ddots & & \ddots & & \beta_\mu \\
\vdots & \ddots & & & & \ddots & & \ddots & \vdots \\
\beta_{\nu+1} & \cdots & \beta_\mu & & & & \beta_0 & \cdots & \beta_\nu
\end{array}\right).
$$

Here $\{\alpha_i\}_{i=0}^{\mu}$ and $\{\beta_i\}_{i=0}^{\mu}$ are the coefficients in (3). We note that $s(A)$ and $s(B)$ are the Strang circulant preconditioners of the Toeplitz parts of A and B respectively, see [3].

Now we are going to prove that S is invertible provided that the given BVM is *stable* and the eigenvalues of J_m are in the negative half of the complex plane $\mathbb{C}$. The stability of a BVM is closely related to two characteristic polynomials defined as follows:

$$\rho(z) = z^{\nu} \sum_{j=-\nu}^{\mu-\nu} \alpha_{j+\nu} z^j \quad \text{and} \quad \sigma(z) = z^{\nu} \sum_{j=-\nu}^{\mu-\nu} \beta_{j+\nu} z^j.$$

Note that they are μ-degree polynomials.

DEFINITION 3.1. *[2, p.101] Given a BVM with the characteristic polynomials $\rho(z)$ and $\sigma(z)$, the region*

$$\mathcal{D}_{\nu,\mu-\nu} \equiv \{q \in \mathbb{C} : \rho(z) - q\sigma(z) \textit{ has } \nu \textit{ zeros inside } |z| = 1 \\ \textit{and } \mu - \nu \textit{ zeros outside } |z| = 1\}$$

is called the region of $A_{\nu,\mu-\nu}$-stability *of the given BVM. Moreover, the BVM is said to be* $A_{\nu,\mu-\nu}$-stable *if the negative half plane is in $\mathcal{D}_{\nu,\mu-\nu}$, i.e.*

$$\mathbb{C}^- \equiv \{q \in \mathbb{C} : Re(q) < 0\} \subseteq \mathcal{D}_{\nu,\mu-\nu}.$$

THEOREM 3.2. *If the BVM for (2) is $A_{\nu,\mu-\nu}$-stable and $h\lambda_k(J_m) \in \mathcal{D}_{\nu,\mu-\nu}$ where $\lambda_k(J_m)$ $(k = 1, \cdots, m)$ are the eigenvalues of J_m, then the preconditioner $S = s(A) \otimes I_m - hs(B) \otimes J_m$ is nonsingular.*

Proof. Since $s(A)$ and $s(B)$ are circulant matrices, their eigenvalues are respectively given by

$$g_A(z) \equiv \alpha_\mu z^{\mu-\nu} + \ldots + \alpha_\nu + \alpha_{\nu-1}\frac{1}{z} + \ldots + \alpha_0 \frac{1}{z^\nu} = \frac{\rho(z)}{z^\nu}$$

and

$$g_B(z) \equiv \beta_\mu z^{\mu-\nu} + \ldots + \beta_\nu + \beta_{\nu-1}\frac{1}{z} + \ldots + \beta_0 \frac{1}{z^\nu} = \frac{\sigma(z)}{z^\nu},$$

evaluated at $\omega_j = e^{2\pi i j/(s+1)}$ for $j = 0, \ldots, s$, see [3]. The eigenvalues $\lambda_{jk}(S)$ of S are therefore given by

$$\lambda_{jk}(S) = g_A(\omega_j) - h\lambda_k(J_m) g_B(\omega_j), \quad j = 0, \cdots, s, \quad k = 1, \cdots, m.$$

We note that if $h\lambda_k(J_m) \in \mathcal{D}_{\nu,\mu-\nu}$, the μ-degree polynomial $\rho(z) - h\lambda_k(J_m)\sigma(z)$ has no roots on the unit circle $|z| = 1$. Thus for all $k = 1, \cdots, m$,

$$g_A(z) - h\lambda_k(J_m) g_B(z) = \frac{1}{z^\nu}\{\rho(z) - h\lambda_k(J_m)\sigma(z)\} \neq 0, \quad \forall |z| = 1.$$

It follows that $\lambda_{jk}(S) \neq 0$ for all $j = 0, \cdots, s$, and $k = 1, \cdots, m$. ☐

COROLLARY 3.3. *If the BVM is $A_{\nu,\mu-\nu}$-stable and $\lambda_k(J_m) \in \mathbb{C}^-$ $(k = 1, \cdots, m)$, then the preconditioner S is nonsingular.*

Proof. We just note that $h\lambda_k(J_m) \in \mathbb{C}^- \subseteq \mathcal{D}_{\nu,\mu-\nu}$. ☐

4. Convergence Rate and Operation Cost. In this section, we show that the spectrum of the preconditioned system is clustered around 1.

THEOREM 4.1. *All the eigenvalues of the preconditioned matrix $S^{-1}M$ are 1 except for at most $2m\mu$ outliers.*

Proof. Let $E = M - S$. We have by (6) and (7),

$$E = (A - s(A)) \otimes I_m - h(B - s(B)) \otimes J_m = L_A \otimes I_m - hL_B \otimes J_m.$$

It is easy to check that L_A and L_B are $(s+1)$-by-$(s+1)$ matrices with nonzero entries only in the following four corners:

- ν-by-$(\mu+1)$ block in the upper left;
- ν-by-ν block in the upper right;
- $(\mu-\nu)$-by-$(\mu+1)$ block in the lower right;
- $(\mu-\nu)$-by-$(\mu-\nu)$ block in the lower left.

Since $\mu > \nu$, rank $(L_A) \le \mu$ and rank $(L_B) \le \mu$. Thus, we have

$$\operatorname{rank}(L_A \otimes I_m) = \operatorname{rank}(L_A) \cdot m \le m\mu$$

and

$$\operatorname{rank}(L_B \otimes J_m) = \operatorname{rank}(L_B) \cdot m \le m\mu.$$

Therefore

$$S^{-1}M = I_{m(s+1)} + S^{-1}E = I_{m(s+1)} + L,$$

where the rank of L is at most $2m\mu$. $\square$

In view of the theorem, we therefore expect that if the Krylov subspace methods, such as the GMRES method [5], is applied to solving the preconditioned system $S^{-1}M\mathbf{x} = S^{-1}\mathbf{b}$, the convergence will be fast, see [4, p.54]. We remark that Bertaccini [1] also showed that with his P-circulant preconditioner P,

$$P^{-1}M = I_{m(s+1)} + U + V,$$

where rank $U \le 2m\mu$, but $\|V\| = O(1)$ (the right hand side of (46) in [1] is bounded by $O(1/(hs)) = O(1)$). Because of the extra term V, we expect the spectrum of $P^{-1}M$ to be less clustered and hence the convergence will be slower. This is verified by the numerical examples in §5.

Regarding the cost per iteration, the main work in each iteration for Krylov subspace methods is the matrix-vector multiplication

$$S^{-1}M\mathbf{z} = (s(A) \otimes I_m - hs(B) \otimes J_m)^{-1}(A \otimes I_m - hB \otimes J_m)\mathbf{z}$$

see Saad [5]. Since A, B are band matrices and J_m is assumed to be sparse, the matrix-vector multiplication $(A \otimes I_m - hB \otimes J_m)\mathbf{z}$ can be done very fast.

To compute $S^{-1}(M\mathbf{z})$, we note that since $s(A)$ and $s(B)$ are circulant matrices, $s(A) = F\Lambda_A F^*$ and $s(B) = F\Lambda_B F^*$, where Λ_A and Λ_B are diagonal matrices containing the eigenvalues of $s(A)$ and $s(B)$ respectively and F is the Fourier matrix, see [3]. It follows that

$$S^{-1}\mathbf{v}(M\mathbf{z}) = (F^* \otimes I_m)(\Lambda_A \otimes I_m - h\Lambda_B \otimes J_m)^{-1}(F \otimes I_m)(M\mathbf{z}).$$

This product can be obtained by using Fast Fourier Transforms and solving s linear systems of order m, see [1] for details. Since J_m is sparse, the coefficient matrices of the m-by-m linear systems will also be sparse. Thus $S^{-1}(M\mathbf{z})$ can be obtained by solving s sparse m-by-m linear systems.

5. Numerical Tests. In this section, we illustrate, by solving the test problems in [1], the efficiency of our proposed preconditioner over other circulant preconditioners. All the experiments are performed in MATLAB with machine precision 10^{-16}. The GMRES [5] and the BICGSTAB [4] methods are employed to solve the preconditioned systems. We use the MATLAB-provided M-files "gmres" and "bicgstab" (see MATLAB on-line documentation) in our implementation. In our tests, the zero vector is the initial guess and the stopping criterion is $||\mathbf{r}_q||_2/||\mathbf{r}_0||_2 < 10^{-6}$, where $\mathbf{r}_q$ is the residual after q iterations.

Example 1. Heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & \\ u(0,t) = u(\pi,t) = 0, & t \in [0, 2\pi], \\ u(x,0) = \sin(x), & x \in [0,\pi]. \end{cases}$$

We discretize the partial differential operator $\partial^2/\partial x^2$ with central differences and step size equals to $\pi/(m+1)$. The system of ODEs obtained is:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = T_m \mathbf{y}(t), & t \in [0, 2\pi], \\ \mathbf{y}(0) = [\sin(x_1), \sin(x_2), \cdots, \sin(x_m)]^t, & \end{cases}$$

where T_m is a scaled discrete Laplacian matrix

$$T_m = \frac{(m+1)^2}{\pi^2} \begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}. \tag{8}$$

The third order GBDF method is used to solve this system of ODEs. The formulae and the additional initial and final equations can be found in [2, p.132].

Example 2. Wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & \\ u(0,t) = u(\pi,t) = 0, & t \in [0, 2\pi], \\ u(x,0) = \sin(x), & x \in [0,\pi], \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & x \in [0,\pi]. \end{cases}$$

We again discretize $\partial^2/\partial x^2$ with central differences and step size equals to $\pi/(m+1)$. The resulting system of ODEs is:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = H_m \mathbf{y}(t), & t \in [0, 2\pi], \\ \mathbf{y}(0) = [\sin(x_1), \sin(x_2), \cdots, \sin(x_{m/2}), 0, \cdots, 0]^t, & \end{cases}$$

where H_m is a Hamiltonian matrix

$$H_m = \begin{pmatrix} 0_{m/2} & I_{m/2} \\ T_{m/2} & 0_{m/2} \end{pmatrix},$$

m	s	I	S	T	P
24	6	9	3	6	6
	12	35	3	7	7
	24	88	3	7	8
	48	176	3	6	8
	96	326	3	6	8
48	6	17	3	6	6
	12	65	3	7	7
	24	166	3	7	8
	48	328	3	7	8
	96	761	3	6	8
96	6	31	3	6	6
	12	129	3	7	7
	24	325	3	7	8
	48	637	3	7	8
	96	1062	3	6	8

m	s	I	S	T	P
24	6	†	5	9	10
	12	>2000	5	9	10
	24	>2000	5	9	10
	48	>2000	5	9	10
	96	>2000	5	9	10
48	6	†	5	9	10
	12	†	5	9	10
	24	†	5	9	10
	48	†	5	9	10
	96	†	5	9	10
96	6	†	5	9	10
	12	†	5	9	10
	24	†	5	9	10
	48	>2000	5	9	10
	96	>2000	5	9	10

TABLE 1

Number of matrix-vector multiplications required for convergence in Example 1 by the GMRES (left) and BICGSTAB (right) methods.

with T_m given in (8). The fourth order ETR_2 method is used to solve this system of ODEs. The formulae and the additional initial and final equations can be found in [2, p.164].

Tables 1 and 2 list the numbers of matrix-vector multiplications $S^{-1}M\mathbf{v}$ required for the convergence of the GMRES and BICGSTAB methods for Examples 1 and 2 with different m and s. The symbol † denotes over- or under-flow in the computation. We remark that we list the numbers of matrix-vector multiplications in place of iteration numbers because then we can compare the effectiveness of the methods by cost. For the GMRES method, only one matrix-vector multiplication with the preconditioned matrix is performed at each iteration. However, there are two matrix-vector multiplications with the preconditioned matrix in each BICGSTAB iteration. We note that since J_m is the discrete Laplacian or the Hamiltonian matrix, it only contains a few diagonals. Therefore, the number of operations per iteration is of $O(ms\log s + ms)$.

In the tables, the notation I means no preconditioner is used, and the symbols S, C and P denote that the Strang, T. Chan and Bertaccini block-circulant preconditioners are used respectively. We see from the tables that when s and/or m increases, the numbers of matrix-vector multiplications required for convergence stay almost the same when circulant preconditioners are used. However, the numbers increase if no preconditioner is used. (The BICGSTAB method even cannot converge after 2000 multiplications.) We also find that the Strang preconditioner performs better than T. Chan's and Bertaccini's preconditioners in all cases.

6. Concluding Remarks. In summary, we have found that the Krylov subspace methods with our proposed preconditioner are efficient methods for solving ODEs. We remark that the idea can be extended readily to nonlinear ODEs. Indeed, by linearization, nonlinear problems can be solved iteratively by solving a discrete system (1) in each step. Thus a fast algorithm for solving (1), such as the one proposed here, will be of greater importance in these cases.

m	s	I	S	T	P
24	6	16	6	7	8
	12	45	6	9	10
	24	60	6	9	9
	48	89	6	9	9
	96	134	5	9	12
48	6	24	6	7	8
	12	79	6	9	10
	24	98	6	9	9
	48	133	6	9	9
	96	182	5	9	12
96	6	44	6	7	8
	12	143	6	9	10
	24	168	6	9	9
	48	224	6	9	9
	96	285	5	9	12

m	s	I	S	T	P
24	6	†	9	9	10
	12	†	9	11	15
	24	>2000	8	13	17
	48	>2000	8	15	16
	96	>2000	8	15	36
48	6	†	9	9	10
	12	†	9	11	15
	24	>2000	8	13	17
	48	>2000	8	15	16
	96	>2000	8	15	36
96	6	†	9	9	10
	12	†	9	11	15
	24	†	8	13	17
	48	>2000	8	15	16
	96	>2000	8	15	36

TABLE 2

Number of matrix-vector multiplications required for convergence in Example 2 by the GMRES (left) and BICGSTAB (right) methods.

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