Application of multigrid techniques to image restoration problems

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ABSTRACT

We briefly describe a multigrid strategy for unilevel and two-level linear systems whose coefficient matrix A_n belongs either to the Toeplitz class or to the cosine algebra of type III and such that A_n can be naturally associated, in the spectral sense, with a polynomial function f. The interest of the technique is due to its optimal cost of $O(N)$ arithmetic operations, where N is the size of the algebraic problem. We remark that these structures arise in certain $2D$ image restoration problems or can be used as preconditioners for more complicated image restoration problems.

Key words: DCT-III matrix algebra, cosine transform, Toeplitz matrices, two-level structures, multigrid and preconditioning.

1. INTRODUCTION

Consider two basic iterative methods

$$x^{(j+1)} = V_n^{(1)} x^{(j)} + \tilde{b}^{(1)} := \mathcal{V}_n^{(1)}(x^{(j)}, \tilde{b}^{(1)}) \quad (1)$$
$$x^{(j+1)} = V_n^{(2)} x^{(j)} + \tilde{b}^{(2)} := \mathcal{V}_n^{(2)}(x^{(j)}, \tilde{b}^{(2)}) \quad (2)$$

for the solution of the linear system $A_n x = b$ where A_n, $M_n^{(i)}$, $V_n^{(i)} := I_n - \left[M_n^{(i)}\right]^{-1} A_n \in \mathbf{C}^{N\times N}$, and b, $\tilde{b}^{(i)} := [M_n^{(i)}]^{-1} b \in \mathbf{C}^N$ with $i = 1, 2$. Given a full-rank matrix $p_n^k \in \mathbf{C}^{N\times K}$, with $K \ll N$, a Two-Grid

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Method (TGM) is defined by the following algorithm[9]:

$$\underline{TGM(V_n^{(1)}, V_n^{(2)}, p_n^k, \nu_1, \nu_2)(x^{(j)})}$$

0. $\tilde{x}^{(j)} = \left[\mathcal{V}_n^{(1)}\right]^{\nu_1} (x^{(j)}, \tilde{b}^{(1)})$
1. $d_n = A_n \tilde{x}^{(j)} - b$
2. $d_k = (p_n^k)^H d_n$
3. $A_k = (p_n^k)^H A_n p_n^k$
4. Solve $A_k y = d_k$
5. $\hat{x}^{(j)} = \tilde{x}^{(j)} - p_n^k y$
6. $x^{(j+1)} = \left[\mathcal{V}_n^{(2)}\right]^{\nu_2} (\hat{x}^{(j)}, \tilde{b}^{(2)})$

The global iteration matrix of $TGM := TGM(V_n^{(1)}, V_n^{(2)}, p_n^k, \nu_1, \nu_2)$ is then given by

$$TGM(V_n^{(1)}, V_n^{(2)}, p_n^k, \nu_1, \nu_2) = \left[V_n^{(2)}\right]^{\nu_2} \left[I_n - p_n^k \left((p_n^k)^H A_n p_n^k\right)^{-1} (p_n^k)^H A_n\right] \left[V_n^{(1)}\right]^{\nu_1}.$$

Steps **1.**→**5.** define the "coarse grid correction" that depends on the projector operator p_n^k, while Steps **0.** and **6.** consist in applying the "intermediate iteration" ν_1 times and the "smoothing iteration" ν_2 times, respectively. Here the names "smoothing iteration" and "intermediate iteration" are in the sense of the multi-iterative methods[12,15]: more precisely, the idea is that the multigrid technique is a multi-iterative procedure, i.e. it is a method composed by at least two different iterations that have a complementary spectral behaviour. In our proposal, we set three basic iterations: (I1) the coarse-grid correction, (I2) the smoothing iteration and (I3) the intermediate iteration. The coarse-grid correction (I1) is a non convergent iteration, but if the operator p_n^k is chosen carefully then the coarse-grid correction shows a very good convergence behaviour in the subspace where A_n is ill-conditioned. The smoothing iteration (I2) is highly convergent in the subspace where A_n is well conditioned, even if it is globally slowly convergent. Finally, the choice of the intermediate iteration (I3) is made in such a way that it is highly convergent in the subspace where the combination of the "smoothing iteration" and of the coarse-grid correction resulted to be less effective.

In conclusion, each of the basic iterations is not effective alone, but their combination is a very fast iterative method.[12,15] Finally, the technique is a real "Multigrid Method" if the solution of the reduced linear system in **4.** is performed again by using a TGM scheme, unless the actual dimension K is so small that the solution of the related linear system can be achieved through a standard direct solver. In what follows, we assume that A_n is sparse in such a way that the cost of multiplying a matrix by a vector is linear as the dimension, the goal being to obtain a multigrid strategy whose cost is linear as the dimension N. Therefore, in practice and in theory, the key point is to define a class of projectors p_n^k and a class of basic iterations (1) and (2) such that the following essential (and basic) constraints are satisfied:

[**a**] assuming $K \leq \theta N$, $\theta \in (0,1)$ independent of N, we should require that the matrix vector product involving p_n^k costs $O(N)$ arithmetic operations; each iteration in (1) and (2) should cost $O(N)$ arithmetic operations;

[**b**] the projected matrix $A_k = (p_n^k)^H A_n p_n^k$ must belong to the same class (of smaller dimension!) as A_n and the computation of its representation has to cost $O(N)$ arithmetic operations at most;

[**c**] the iterative multigrid procedure should have an optimal convergence rate, i.e., the number of iterations in order to reach a preassigned accuracy must remain bounded by a constant independent of N.

The constraint [**c**] implies that the overall complexity of the method is proportional to the cost of a single multigrid sweep (since the number of iterations is bounded by a constant). The requirement reported in [**b**] that "A_k has the same structural properties of A_n" is a really basic requirement; otherwise it would be meaningless to talk about recursion and multigrid. Finally, the statements in [**a**] and the fact that the cost of computing a representation of A_k is linear as the dimension have the nice consequence that the cost of a single

multigrid iteration is globally linear as the dimension of A_n (if we use $c \log N$ levels with $c > 0$ and independent of N).
In conclusion, the features reported in [**a**], [**b**] and [**c**] would imply that the proposed method is optimal in the sense of Axelsson and Neytcheva,[1] since the "inverse" problem of solving a linear system with coefficient matrix A_n is asymptotically of the same cost of the "direct" problem of multiplying A_n by a vector.

The paper is organized as follows. In Section 2 we report definitions and properties concerning the algebra of DCT III matrices and the class of Toeplitz matrices. In Section 3 we define our multigrid technique for unilevel DCT III matrices and we discuss on how to implement requirements [**a**], [**b**] and [**c**]. Section 4 is devoted to the extension of the technique to the two-level case and to a brief discussion on the Toeplitz case. Finally, Section 5 deals with numerical experiments related to some image restoration applications.

2. UNILEVEL TOEPLITZ AND DCT III MATRICES

Let f be a real valued even trigonometric polynomial of degree c defined over the interval $\Omega = (0, 2\pi]$. From the Fourier coefficients of f, i.e. $a_j = (2\pi)^{-1} \int_\Omega f(x) e^{-\hat{\mathbf{i}} jx}\, dx$, $\hat{\mathbf{i}}^2 = -1$, $j \in \mathbf{Z}$, we define the sequence of Toeplitz matrices $\{T_n(f)\}$, where $T_n(f) = \{a_{t-s}\}_{s,t=1}^n \in \mathbf{C}^{n \times n}$ is said to be the Toeplitz matrix of order n generated by f. It is clear that $a_j = 0$ if $|j| > c$. Moreover, since f is even, i.e. $f(x) = f(|x|)$, it follows that f is completely determined by its values on the sub-interval $[0, \pi]$ and in addition $a_j = a_{-j} \in \mathbf{R}$. More explicitly, the matrix $T_n(f)$ can be conveniently rewritten in terms of Jordan blocks, that is $T_n(f) = \sum_{|j| \le c} a_j J_n^{[j]}$, where $J_m^{[l]}$ denotes the matrix of order m whose (s,t) entry equals 1 if $t - s = l$ and equals zero otherwise ($J_m^{[l]}$ is the l-th power of the basic Jordan block if $l \ge 1$, is the identity matrix if $l = 0$ and coincides with the $|l|$-th power of transposed Jordan block if $l \le -1$). On the other hand, the DCT III matrix of order n generated by the same polynomial f is defined as $S_n(f) = Q_n D_f^{[n]} Q_n^T$, where $D_f^{[n]} = \mathrm{diag}_{0 \le j \le n-1} f(x_j^{[n]})$ with $x_j^{[n]} = \pi j / N$ and the matrix

$$Q_n = \left[\sqrt{\frac{2 - \delta_{j,1}}{n}} \cos\left(\frac{(i-1)(2j-1)\pi}{2n} \right) \right]_{i,j=1}^n \qquad \delta_{1,1} = 1, \delta_{j,1} = 0 \text{ if } j \neq 1,$$

denotes the unitary matrix diagonalizing the considered algebra, i.e. the DCT III transform matrix. The matrix $S_n(f)$ is the natural preconditioner in the DCT III algebra of the corresponding Toeplitz matrix $T_n(f)$ and it can be considered the analog of the Strang preconditioner[5] in the circulant algebra and of the natural preconditioner in the τ algebra.
The relationships between $S_n(f)$ and $T_n(f)$ are even stronger (see[4]), since $S_n(f) = T_n(f) + H_n(f)$, where $H_n(f)$ is a special centrosymmetric Hankel matrix generated by f. A Hankel matrix is one whose entries are constant along any lower-left–upper-right diagonal. As with Toeplitz matrices, one can consider Hankel matrices generated by a symbol f over Ω. With the same notations, let $\{a_j\}$ denote the Fourier coefficients of f. Then we have $H_n(f) = \sum_{1 \le |j| \le c} a_j K_n^{[j]}$, where $K_m^{[l]}$ denotes the matrix of order m whose (i,j) entry equals 1 if $i + j = (l+1) \bmod 2n$ and equals zero otherwise.
It is evident that $H_n(f)$ is a low rank correction for every polynomial f, so that $\{S_n(f)\}$ and $\{T_n(f)\}$ are equally distributed and they share the same asymptotic spectral features. This is in essence the reason for which it is reasonable to expect that $S_n(f)$ is a good preconditioner for $T_n(f)$.

3. MULTIGRID METHOD FOR UNILEVEL DCT III MATRICES

In this section we describe our multigrid proposal for DCT III matrices: in particular we give a constructive procedure for defining the basic iterations and the projectors in such a way that requirements [**a**], [**b**] and [**c**] are satisfied.

3.1. Structural and algebraic requirements: [a] and [b]

Let us consider $A_n = S_n(f)$ unilevel DCT III matrix generated by a univariate trigonometric polynomial f. The most important requirement is the definition of a class of projectors p_n^k such that "A_n belonging to the

DCT III algebra of size n" implies that "$A_k = (p_n^k)^H A_n p_n^k$ belongs to the DCT III algebra of size k"; otherwise it makes no sense to define a recursive strategy as the multigrid one.

Given a DCT III matrix P_n of size N (with $N = n$ in the unilevel case), we define our projectors p_n^k as $P_n T_n^k$ where the operator $T_n^k \in \mathbf{R}^{N \times K}$, $n = 2k$, is defined as

$$(T_n^k)_{i,j} = \begin{cases} 1 & \text{for } \ i \in \{2j-1, 2j\}, \ j = 1, \ldots, K, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The operator T_n^k represents a spectral link between the space of the frequencies of size n and the corresponding space of frequencies of size k according to the following Lemma.

LEMMA 3.1.[4] *The following representation holds true:*

$$\left[T_n^k\right]^T Q_n = Q_k[D_k, S_k] \tag{4}$$

where Q_m is the unilevel DCT-III transform matrix of size m and T_n^k is the operator defined in (3). *Moreover,* $S_k = \Pi_k \tilde{D}_k$, $D_k = \mathrm{diag}_{s=1,\ldots,K}[\sqrt{2}\cos((s-1)\pi/(4K))]$, $\tilde{D}_k = \mathrm{diag}_{s=1,\ldots,K}[-\sqrt{2}\cos((s-1)\pi/(4K)+\pi/4)]$ *and* $(\Pi_k)_{s,t} = 1$ *if and only if* $(s+t) \pmod{K} = 2$ *and* $(s,t) \neq (1,1)$. It is worth stressing that the quoted choice of T_n^k corresponds to the sum of the two analogous structures just considered in the τ case[7,8] and in the circulant case[14,15] for constructing efficient multigrid procedures.
Notice also that the simple relation stated in the previous Lemma is the key step in defining a multigrid method, since it allows us to obtain again a DCT III matrix at the lower level.

PROPOSITION 3.2. *Let A_n and P_n be two DCT III matrices of size N and let $p_n^k = P_n T_n^k$, $n = 2k$. Then, the matrix $A_k = (p_n^k)^T A_n p_n^k$ is a DCT III matrix of size K. In addition, if A_n is positive definite and P_n is invertible, then A_k is positive definite.*

Proof. The projected matrix $(p_n^k)^T A_n p_n^k$ can be spectrally decomposed by taking into account relation (4). Indeed, setting $A_n = Q_n D_{A_n} Q_n^T$ and $Q_n D_{P_n} Q_n^T$, we have

$$\begin{aligned} (p_n^k)^T A_n p_n^k &= \left[T_n^k\right]^T P_n A_n P_n T_n^k \\ &= \left[T_n^k\right]^T Q_n D_{P_n^2 A_n} Q_n^T T_n^k \\ &= Q_k \left(D_k D_{1,P_n^2 A_n} D_k + \Pi_k \tilde{D}_k D_{2,P_n^2 A_n} \tilde{D}_k \Pi_k \right) Q_k^T \end{aligned}$$

where we pose

$$D_{P_n^2 A_n} = \begin{bmatrix} D_{1,P_n^2 A_n} & \\ & D_{2,P_n^2 A_n} \end{bmatrix}.$$

Therefore, it is evident that the matrix $\Delta_k = D_k D_{1,P_n^2 A_n} D_k + \Pi_k \tilde{D}_k D_{2,P_n^2 A_n} \tilde{D}_k \Pi_k$ is a diagonal matrix and finally, $A_k = Q_k \Delta_k Q_k$ is, by construction, a DCT III matrix of size K.
The invertibility of P_n and the fact that T_n^k is full rank implies that p_n^k has also full rank K and therefore, taking into account the positive definiteness of A_n, we deduce that A_k is positive definite as well. ☐

As the last step of this subsection, we define the iterations to be used in Steps **0.** and **6.** A reasonable choice is to set $V_n = I_n - \omega A_n$ where $\omega = 2\|f\|_\infty^{-1}$ for the intermediate iteration and $\omega = \|f\|_\infty^{-1}$ for the smoothing step. An alternative good proposal consists in applying the ordinary conjugate gradient method as intermediate iteration and the Gauss Seidel method as smoothing iteration. Finally, let us check carefully the requirements [**a**] and [**b**] with regard to the proposed definitions of p_n^k and of the basic iterations. The parameter $\theta = 1/2$ is in the interval $(0,1)$, the cost of a matrix vector product involving p_n^k is $O(N)$, if the DCT III matrix P_n is banded for some bandwidth independent of n. The computation of the representation of A_k can be done formally in $O(1)$ operations since we know that A_k belongs to the algebra (see section 12 in[13]). Finally, all the proposed basic iterations have a linear cost as the dimension since we have assumed, at the very beginning, that A_n is banded. In conclusion, in what follows also the matrix P_n has to be taken banded.

3.2. Analytical requirements: [b] and [c]

Here we will consider the convergence requirements of [**c**] and some further stuctural requirements of [**b**] that possess a more analytical flavour. Indeed, it is important to preserve the "structure" also in a stronger sense, i.e. in applying the full multigrid procedure to $A_n := S_n(f)$ with f nonnegative even polynomial, we have to require that the matrix A_k, obtained at the lower level, is DCT III of size K, and of the same type. More precisely, we have to require that A_k can be viewed as $S_k(\hat{f})$, where $\hat{f}$ is a nonnegative even polynomial with the same essential features of f.

PROPOSITION 3.3.[4] *Let $n = 2k$, $p_n^k = S_n(p)T_n^k$, and $A_n = S_n(f)$. Suppose that f and p are even polynomials with f being nonnegative. Then, the matrix $A_k = (p_n^k)^T S_n(f) p_n^k$ coincides with $S_k(\hat{f})$ where $\hat{f}$ is an even polynomial of the form $\hat{f}(x) = 2[\cos^2(x/4) f(x/2) p^2(x/2) + \sin^2(x/4) f(\pi - x/2) p^2(\pi - x/2)]$, with $x \in [0, \pi]$.*

Now, the requirements dictated by the convergence results come into the play, so that we describe conditions on the polynomial p that will insure the optimal convergence rate of the multigrid procedure. More specifically, if f has a unique zero $x^0 \in [0, \pi]$, then we consider $\hat{x}^0 = \pi - x^0$ and we set $P_n = S_n(p)$, where p is the even trigonometric polynomial defined as

$$p(x) = (2 - 2\cos(x - \hat{x}^0))^{\lceil \beta/2 \rceil} \sim |x - \hat{x}^0|^{2\lceil \beta/2 \rceil} \quad \text{over } [0, \pi] \tag{5}$$

with

$$\beta \;\geq\; \beta_{\min} = \min \left\{ i \,\middle|\, \lim_{x \to x^0} \frac{\sin^2(x/2)}{\cos^2(x/2)} \frac{|x - x^0|^{2i}}{f(x)} = 0 \right\}, \tag{6}$$

$$0 \;<\; p^2(x) + p^2(\pi - x). \tag{7}$$

If f has more than one zero in $[0, \pi]$, then the corresponding polynomial p will be the product of the basic polynomials of kind (5), satisfying the condition (6) for every single zero and globally the condition (7). The quoted choice of the polynomial p allows us to describe more precisely further properties of the function $\hat{f}$, generating the DCT-III matrix $A_k = S_k(\hat{f})$, that are essential in a recursive application of the multigrid procedure.

PROPOSITION 3.4.[4] *Let $n = 2k$, $p_n^k = S_n(p)T_n^k$, and $A_n = S_n(f)$. Suppose that f and p are even polynomials with f being nonnegative. Assume that β is a fixed constant independent of n and that p globally satisfies the previously described conditions (5)–(7). Then,*

1. *if f is a polynomial then $\hat{f}$ is a polynomial with a fixed degree only depending on the orders of the zeros of f;*
2. *if x^0 is a zero of $f(x)$, then $y^0 = 2x^0$ is a zero of $\hat{f}$ (clearly if $\pi/2 \leq x^0 \leq \pi$ then y^0 is rewritten as $y^0 = 2(\pi - x^0)$);*
3. *the order of the zero y^0 of $\hat{f}$ is exactly the same as the one of the zero x^0 of f, except in the case $x^0 = \pi$ where the order of the zero y^0 is the one of the zero x^0 of f increased by two.*

Notice that the previous proposition also shows why the computational complexity is not increased at the lower levels. Indeed, it is easy to prove that the degree of the generating functions $f^{[j]}$, $j = 0, \ldots, \log(N) - 1$, of any multigrid iteration stay bounded by a constant which only depends on the first function $f^{[0]}$ since the number of the zeros of each $f^{[j]}$ equals the number of the zeros of $f^{[0]}$ and their orders are increased at most by 2 (by item *3.*). More specifically, if the r-th zero of $f^{[t]}$ is π, then the r-th zero of $f^{[s]}$ with $s < t$ is $\pi/2^{t-s}$ and the r-th zero of $f^{[s]}$ with $s > t$ is 0: this shows that the order of the k-th zero of $f^{[s]}$ is the same as for $f^{[0]}$ for $s < t$ and is simply increased by two if $s > t$. In addition, the crucial informations on the "nature" of $\hat{f}$, namely the position and degree of the zeros of $\hat{f}$, are formaly established in claims *2.* and *3.*, so that at the lower level the new projector is easily defined in the same way.

Finally, mention has to be made to the following condition on the polynomial p

$$\beta \geq \beta_{\min} = \min \left\{ i \,\middle|\, \lim_{x \to x^0} \frac{\sin^2(x/2)}{\cos^2(x/2)} \frac{|x - x^0|^{2i}}{f(x)} < \infty \right\}. \tag{8}$$

This condition is slightly weaker than (6) and, in connection with (5) and (7), is equivalent to the optimality of the two-grid method, but it is not sufficient in general for the optimality of the full multigrid method. For instance, for the discretization of the Laplacian with homogeneous Neumann boundary conditions, in the light of (6) we have $f(x) = 2 - 2\cos(x)$, $x^0 = 0$, $\deg(f) = 1$ and then we can chose $\deg(p) = 0$. This means that the corresponding matrix $S_n(f)$ is singular, but its one-rank correction $\tilde{S}_n(f) = S_n(f) + \delta ee^T$, $e_i = 1$, $i = 1, \ldots, N$, is invertible. Moreover, we do not need a nontrivial weight function p in order to satisfy (8) and therefore, we can simply choose $p_n^k = T_n^k$ for devising an optimal two-grid method: the effectiveness of this quite strange conclusion is confirmed experimentally in Section 5: we refer the reader to the first column of Table 1 and to the first column of Table 2 for the optimality of the two-grid method and the non optimality of the full multi-grid method ((8) holds true, but (6) is violated).

4. THE TWO-LEVEL CASE

Let f be a 2-variate even trigonometric polynomial defined over the square Ω^2, with $\Omega = (0, 2\pi]$ and having degree $c = (c_1, c_2)$, $c_r \geq 0$ with regard to the variables $x = (x_1, x_2)$. From the Fourier coefficients of f, i.e. $a_j = (2\pi)^{-2} \int_{\Omega^2} f(x) e^{-\hat{\mathbf{i}}(j,x)}\, dx$, $\hat{\mathbf{i}}^2 = -1$, $j = (j_1, j_2) \in \mathbf{Z}^2$ with $(j, x) = j_1 x_1 + j_2 x_2$, we build the sequence of two-level Toeplitz matrices $\{T_n(f)\}$, $n = (n_1, n_2)$, where $T_n(f) = \{a_{t-s}\}_{s,t=e^T}^{n} \in \mathbf{C}^{N(n) \times N(n)}$, $N(n) = n_1 n_2$, $e = (1, 1)^T \in \mathbf{N}^2$, is said to be the Toeplitz matrix of order N generated by f. It is clear that $a_j = 0$ if the condition $|j| \leq c$ is violated (i.e. if there exists $\bar{r}$ such that the absolute value of $j_{\bar{r}}$ exceeds $c_{\bar{r}}$). Moreover, since f is even, i.e. $f(x) = f(|x|)$ with $|x| = (|x_1|, |x_2|)$, it follows that f is completely determined by its values on $[0, \pi]^2$ and in addition $a_j = a_{-j} \in \mathbf{R}$. More explicitly, the matrix $T_n(f)$ can be conveniently rewritten in terms of Jordan blocks, that is $T_n(f) = \sum_{|j| \leq c} a_j J_n^{[j]} = \sum_{|j_1| \leq c_1} \sum_{|j_2| \leq c_2} a_{(j_1, j_d)} J_{n_1}^{[j_1]} \otimes J_{n_2}^{[j_2]}$, where, in the above equation, $\otimes$ denotes tensor product and $J_m^{[l]}$ with scalar l and m is the same structure considered in Section 2. Indeed, the set $\{J_n^{[j]}\}$ is the canonical basis of the linear space of the two-level Toeplitz matrices of partial dimensions n_1 and n_2. On the other hand, the two-level matrix belonging to the cosine III algebra of order $N(n)$ generated by the same polynomial f is defined as $S_n(f) = Q_n D_f^{[n]} Q_n^T$, where the matrix $Q_n = Q_{n_1} \otimes Q_{n_2}$ denotes the unitary matrix diagonalizing the considered algebra, i.e. the 2 level DCT III transform matrix and $D_f^{[n]} = \mathrm{diag}_{\,0 \leq j \leq n - e^T} f(x_j^{[n]})$. Here, the relation $0 \leq j \leq n - e^T$ and the expression $x_j^{[n]} = \pi j / n = (\pi j_1 / n_1, \pi j_2 / n_2)$ are intended to be componentwise. As in the unilevel setting, we observe a Toeplitz plus Hankel representation,[4] i.e. $S_n(f) = T_n(f) + H_n(f)$, where $H_n(f) = \sum_{e \leq |j| \leq c} a_j K_n^{[j]} = \sum_{1 \leq |j_1| \leq c_1} \sum_{1 \leq |j_2| \leq c_2} a_{(j_1, j_2)} K_{n_1}^{[j_1]} \otimes K_{n_2}^{[j_2]}$.

4.1. The requirements [a], [b] and [c] in the two-level case

We first observe that the definition of the basic iterations can be done verbatim in the multilevel case. The cost will result again linear as the dimension $N = N(n)$. Therefore, the only delicate point concerns the definition of the projector p_n^k and the consequences on the structural and analytical properties of the projected matrix A_k. Therefore, we will focus on this point, by considering the use of tensorial arguments as main tool. More precisely, the projector is constructed as $p_n^k = P_n U_n^k$, where P_n is a two-level DCT-III matrix with $n = (n_1, n_2)$ and the operator U_n^k is defined as $T_{n_1}^{k_1} \otimes T_{n_2}^{k_2}$ with $n_r = 2k_r$ and $T_{n_r}^{k_r}$ being the unilevel operator given in equation (3).
As in the unilevel case the key step is in highlightining the link between the space of frequencies of sizes $n = (n_1, n_2)$ and the corresponding space of frequencies of sizes $k = (k_1, k_2)$. From the definition of U_n^k and from equation (4), it holds that

$$\begin{aligned} \left[U_n^k\right]^T Q_n &= (T_{n_1}^{k_1} \otimes T_{n_2}^{k_2})^T (Q_{n_1} \otimes Q_{n_2}) \\ &= [(T_{n_1}^{k_1})^T Q_{n_1}] \otimes [(T_{n_2}^{k_2})^T Q_{n_2}] \end{aligned}$$

$$
\begin{aligned}
&= [Q_{k_1}[D_{k_1}, S_{k_1}]] \otimes [Q_{k_2}[D_{k_2}, S_{k_2}]] \\
&= Q_k[[D_{k_1}, S_{k_1}] \otimes [D_{k_2}, S_{k_2}]]
\end{aligned}
$$

where $Q_k = Q_{k_1} \otimes Q_{k_2}$. By using the former relation it is easy to prove that A_k is still a DCT III matrix of order $k = (k_1, k_2)$ if P_n is chosen as a DCT III matrix of order $n = (n_1, n_2)$.
In analogy with the unilevel case, for computational reasons (requirements [**a**] and [**b**]), we restrict the choice of the DCT-III matrix P_n to the case where $P_n = S_n(p)$ with p an even trigonometric polynomial. In addition, in order to get optimal convergence rate (see requirement [**c**]), we will impose the following conditions. If the function f has a unique zero $x^0 \in [0, \pi]^2$, then we set $P_n = S_n(p)$, where p is the polynomial defined over $[0, \pi]^2$ as

$$
p(x) \sim \prod_{\hat{x}^0 \in M(x^0)} \left(\sum_{r=1}^{2} |x_r - \hat{x}_r^0|^{2\lceil \beta/2 \rceil} \right) \tag{9}
$$

where

$$
\beta \quad \geq \quad \beta_{\min} = \min \left\{ i | \sum_{r=1}^{2} \lim_{x_r \to x_r^0} \frac{\sin^2(x_r/2)}{\cos^2(x_r/2)} \frac{|x_r - x_r^0|^{2i}}{f(x)} = 0 \right\}, \tag{10}
$$

$$
0 \quad < \quad \sum_{\hat{x} \in M(x) \cup \{x\}} p^2(\hat{x}) \tag{11}
$$

with $M(x)$ being the set of the "mirror points" of x. A formal definition is the following: $\hat{x} \in M(x)$ if and only if $\hat{x} \neq x$ and for any $r = 1, 2$ it holds $\hat{x}_r \in \{x_r, \pi - x_r\}$. In the unilevel setting, it is evident that the unique mirror point of x is $\pi - x$, while in the two-level context we observe three mirror points. Notice that for every $\hat{x} \in M(x)$ we have $M(\hat{x}) = \{M(x) \backslash \{\hat{x}\}\} \cup \{x\}$. If f has more than one zero in $[0, \pi]^2$, then the corresponding polynomial p will be the product of the basic polynomials of kind (9), satisfying the condition (10) for all zeros and globally the condition (11). The quoted choice of p induces some useful properties on the function $\hat{f}$ generating the DCT-III matrix A_k at the lower level.

Proposition 4.1.[4] *Let $n = 2k$, $p_n^k = S_n(p) U_n^k$, and $A_n = S_n(f)$. Suppose that f and p are even polynomials with f being nonnegative. Assume that β is a fixed constant independent of n and that p globally satisfies the previously described conditions (9)–(11). Then,*

1. *the matrix $(p_n^k)^T S_n(f) p_n^k$ coincides with $S_k(\hat{f})$ where $\hat{f}$ is the even nonnegative polynomial described by the formula $\hat{f}(x) = 4 \left[f(x/2) p^2(x/2) C^2(x/4) + \sum_{y \in M(x/2)} f(y) p^2(y) C^2(y/2) \right]$ for $x = (x_1, x_2) \in [0, \pi]^2$ and where $C(x) = \prod_{r=1}^{2} \cos(x_r)$. If f is a polynomial then $\hat{f}$ is a polynomial with a fixed degree only depending on the orders of the zeros of f.*
2. *If x^0 is a zero of $f(x)$ then $y^0 = 2x^0$ is a zero of $\hat{f}$ (clearly if $\pi/2 \leq x_r^0 \leq \pi$ then y_r^0 is rewritten as $y_r^0 = 2(\pi - x_r^0)$).*
3. *The order of the zero y^0 of $\hat{f}$ is exactly the same as the one of the zero x^0 of f if $x_r^0 \neq \pi$ for all $r = 1, 2$. If $x_r^0 = \pi$ for a given r then the order of the zero y^0 is the one of the zero x^0 of f increased by two.*

4.2. The Toeplitz case

The structure of the proposed multigrid technique is essentially the same as in the DCT-III algebra case (for any details refer to[7,8,13]). In the unilevel setting we define the matrix $T_n^k \in \mathbf{R}^{N \times K}$, $n = 2k + 1$, with

$$
(T_n^k)_{i,j} = \begin{cases} 1 & \text{for } \ i = 2j, \quad j = 1, \ldots, K, \\ 0 & \text{otherwise} \end{cases}
$$

and their variations $\tilde{T}_n^k[t], T_n^k[t]$ (see[6,13]), that are employed in order to preserve the exact Toeplitz structure at each subsequent level of projection. More precisely, for every $t \geq 0$, $\tilde{T}_n^k[t]$ coincides with the submatrix of T_n^k

obtained by deleting its first and last t columns, i.e., by setting $0_\alpha^\beta \in \mathbb{R}^{\alpha\times\beta}$ the null matrix,

$$\tilde{T}_n^k[t] = \begin{bmatrix} 0_{2t}^{k-2t} \\ T_{n-4t}^{k-2t} \\ 0_{2t}^{k-2t} \end{bmatrix} \in \mathbf{R}^{N\times(K-2t)}$$

However, a preliminary numerical experimentation proved that the convergence behavior is no more optimal (as for T_n^k), while the optimality is preserved by considering a matrix of the form

$$T_n^k[t] = \begin{bmatrix} 0_{t}^{k-t} \\ T_{n-2t}^{k-t} \\ 0_{t}^{k-t} \end{bmatrix} \in \mathbf{R}^{N\times(K-t)}.$$

The projectors are defined as $P_n T_n^k[t]$, where P_n is the Toeplitz matrix generated by a suitable nonnegative trigonometric polynomial of degree b (see[7,13]) and $t = b-1$ is the minimal integer such that $[T_n^k[t]]^T P_n A_n P_n T_n^k[t]$ is Toeplitz when A_n is Toeplitz. Analogously, in the two-level case, the matrix P_n equals $T_n(p)$ where $n = (n_1, n_2)$ and p is a suitable bivariate nonnegative polynomial of partial degrees t_1 and t_2 (see[7,13]). Therefore, we set $U_n^k[t] = T_{n_1}^{k_1}[t_1] \otimes T_{n_2}^{k_2}[t_2]$, $p_n^k = P_n U_n^k[t]$, $t = (t_1, t_2)$ and so the projected matrix takes the form $[U_n^k[t]]^T P_n A_n P_n U_n^k[t] \in \mathbb{R}^{\hat{k}_1\times\hat{k}_2}$ where $\hat{k}_1 = (n_1 - 2t_1 - 1)/2$ and $\hat{k}_2 = (n_2 - 2t_2 - 1)/2$.
The definition of the smoothing operators follows the same lines as in the preceeding sections and will not be discussed explicitly.

5. NUMERICAL EXPERIMENTS AND CONCLUSIONS

The numerical experiments will concern unilevel and two-level (banded) DCT-III/Toeplitz linear systems with generating functions having zeros at $x^0 \in \{0, \pi\}$, the interest being related to problems in imaging. More precisely, we consider linear systems coming from a super-resolution problem and linear systems coming from the image restoration context in which we suppose that the blur operator is compactly supported and spatially invariant. In both cases we consider reflecting (Neumann) boundary conditions (refer e.g. to[2,10]) that give raise to DCT-III structures and Dirichlet boundary conditions giving raise to Toeplitz structures. We recall that in the case of DCT III matrices we always use Richardson iterations with $\omega = 2\|f\|_\infty^{-1}$ and $\omega = 1\|f\|_\infty^{-1}$ as intermediate iteration and smoothing iteration respectively. In the case of Toeplitz systems we prefer the alternative choice of the conjugate gradient and of the Gauss Seidel iteration. Moreover the choice of the projectors is performed according to the analysis of the previous sections and, finally, in all the tables the vector x_e denotes the exact solution of the related linear system.

Super-resolution problem
In this setting we consider Neumann boundary conditions and consequently one has to solve linear systems of the form $A_n x = b$, where A_n is a two-level DCT III matrix of separable type. More specifically, we have $A_n = S_n(p(x_1)p(x_2))$ with $p(x_i) = 2 + 2\cos(x_i)$, $n = (n_1, n_2)$ and therefore, $A_n = S_{n_1}(p) \otimes S_{n_2}(p)$. Due to the tensorial structure of the linear algebra problem, it is sufficient to provide an efficient numerical procedure for the unilevel problem in order to solve the two-level problem too: some numerical evidences concerning this basic unilevel DCT-III problem are reported in Subsection 5.2. Finally, if the boundary conditions are of Dirichlet type, then we have a Toeplitz structure with $A_n = T_{n_1}(p) \otimes T_{n_2}(p)$.

Image-restoration problem
In the following we let $A_n(\cdot)$ to be either $S_n(\cdot)$ or $T_n(\cdot)$. Let $\mathcal{S}$ be the true image (for instance a "satellite") and let us consider the blurred image

$$\mathcal{S}_q = A_n(\psi(x_1, x_2)[4 + 2\cos(x_1) + 2\cos(x_2)]^q)\mathcal{S} \tag{12}$$

where the matrix $A_n(\psi(x_1, x_2)[4 + 2\cos(x_1) + 2\cos(x_2)]^q)$ represents the compactly supported and spatially invariant "blurring operator". Here $[4 + 2\cos(x_1) + 2\cos(x_2)]^q$ has a zero at (π, π) of order $2k$, its Fourier coefficients are nonnegative (related to the stencil $[1, 1, 4, 1, 1]$) and $\psi(x_1, x_2)$ is a nonnegative polynomial with

Table 1. Twogrid - $1D$ case: $S_n(f)x = b$, $f(0) = 0$, $(x_e)_i = i/N$.

N	$q=1$		$q=2$		$q=3$		
	$w=0$	$w=1$	$w=1$	$w=2$	$w=1$	$w=2$	$w=3$
16	25	7	15	13	-	34	32
32	26	7	16	15	36	35	34
64	27	7	16	16	36	35	35
128	28	7	16	16	36	35	35
256	28	7	16	16	36	35	35
512	29	7	16	16	36	35	35

Table 2. Multigrid - $1D$ case: $S_n(f)x = b$, $f(0) = 0$, $(x_e)_i = i/N$.

N	$q=1$		$q=2$		$q=3$		
	$w=0$	$w=1$	$w=1$	$w=2$	$w=1$	$w=2$	$w=3$
16	1	1	1	1	1	1	1
32	26	7	16	15	36	34	32
64	60	7	17	16	63	35	34
128	125	7	18	16	123	35	35
256	251	7	18	16	225	35	35
512	497	7	18	16	391	35	35

nonnegative Fourier coefficients. The considered choice is made in such a way that the resulting blur operator is a band approximation of the classical Gaussian blur whose Fourier coefficients are positive, symmetric and decay exponentially and whose generating function is close to zero in a neighborhood of (π, π) and is positive elsewhere. Finally the presence of the term $\psi(x_1, x_2)$ leads to a larger bandwidth so that the resulting blurring effect is more realistic. In Subsection 5.2, we report some numerical evidences concerning the problem of reconstruct $\mathcal{S}$ from $\mathcal{S}_q$ by using our multigrid in the case of the DCT-III algebra and with several (artificial) data sets $\mathcal{S}$. Finally, in Subsection 5.3, we show an instance of the same problem in the Toeplitz case and with a true image of a satellite $\mathcal{S}$. A common point in the two applications is the fact that the generating functions show a zero in π or in (π, π). According to our multigrid theory this means that the reduced matrices will have generating functions with a unique zero at 0 or at $(0, 0)$ respectively: this property will be mantained in all the further levels according to our theoretical analysis. We stress that the latter property is inherent to the discretization of differential problems, while the original problems can be viewed as discretizations of integral problems. Due to this duality, we will focus our attention to the case of generating functions with zeros at 0 or at π in the unilevel context, and at $(0, 0)$ or at (π, π) in the two-level context.

5.1. Case $x^0 = 0$ or $x^0 = (0, 0)$

Here, we consider the solution of linear systems of the form $\tilde{S}_n(f_q)\mathbf{x} = \mathbf{b}$ where $\tilde{S}_n(f_q) = S_n(f_q) + \delta ee^T/N$ and $f_q(x) = [2 - 2\cos(x)]^q$ with a unique zero at $x^0 = 0$ of order $2q$. Notice that, according to Proposition 3.2, the position of the zero at the lower levels is exactly the same as at the first level; consequently the function $p(x)$ in the projectors can be the same at all the subsequent levels. This property is of great help for a simplified implementation of the proposed multigrid algorithm. Both our two grid and multigrid procedures are tested for different values of q and for several choices of the dimension N. Concerning the polynomial $p_w(x) = [2 - 2\cos(\pi - x)]^w$, related to $p_n^k = S_n(p_w)T_n^k$, the choice of w is performed taking into account the condition (6). It is transparent that the lower is the value of w, the greater will be the advantage from a computational viewpoint. The results in Table 1 confirm the optimality of the corresponding two-grid iteration in the sense that the number of iterations is uniformly bounded by a constant not depending on the size N indicated in the first column. Following the suggestions in (6) and (10), in order to have a full multigrid optimality we must choose w at least equal to 1 if $q = 1, 2$ and at least equal to 2 if $q = 3$, as confirmed in

Table 3. Twogrid - $2D$ case: $S_n(f)x = b$, $f(0,0) = 0$, $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1$.

	$q=1$		$q=2$		$q=3$		
$N = n_1 n_2$	$w=0$	$w=1$	$w=1$	$w=2$	$w=1$	$w=2$	$w=3$
32^2	22	16	36	35	75	-	-
64^2	22	16	36	36	75	74	73
128^2	22	16	36	36	75	74	73

Table 4. Multigrid - $2D$ case: $S_n(f)x = b$, $f(0,0) = 0$, $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1$.

	$q=1$		$q=2$		$q=3$		
$N = n_1 n_2$	$w=0$	$w=1$	$w=1$	$w=2$	$w=1$	$w=2$	$w=3$
16^2	1	1	1	-	1	-	-
32^2	22	16	36	1	75	1	1
64^2	52	16	36	36	119	74	73
128^2	108	16	36	36	296	74	73
256^2	217	16	37	36	670	74	73
512^2	430	16	37	36	1329	74	73

Table 2. Analogous considerations can be made in the two-level setting, as reported in Tables 3 and 4, where we consider $f_q(x_1, x_2) = [2 - 2\cos(x_1)]^q + [2 - 2\cos(x_2)]^q$ with a zero at $x^0 = (0,0)$ of order $2q$ and we chose

$$p_w(x_1, x_2) = [(4 - 2\cos(x_1) - 2\cos(\pi - x_2))(4 - 2\cos(\pi - x_1) - 2\cos(x_2))(4 - 2\cos(\pi - x_1) - 2\cos(\pi - x_2))]^w.$$

5.2. Case $x^0 = \pi$ or $x^0 = (\pi, \pi)$

As emphasized at the beginning of the section, it is interesting to consider the case of a univariate generating function having a unique zero in position π and, analogously, the case of a two-variate function possessing a unique zero at (π, π). The remarkable fact is that the choice of the proper projector according to (5)–(7) in the unilevel case produces at the lower level a DCT-III matrix A_k associated with a generating function having a unique zero at 0. The same is true if we replace (8) by (6) and in the multilevel case by imposing either the multilevel generalization of (8) or (10). Therefore, starting from that level, the multigrid strategy is the same as in the previously considered Section 5.1, both from the point of view of practical and theoretical issues. Tables 5 and 6 confirm the optimality of our two-grid and multigrid procedures where the constant number of iterations is very small and seems to be independent of the spectral decomposition of the exact solution. Finally, we point out that the problem in Tables 5 and 6 refers to the image restoration problem with $k = 1$ and $\psi(x) = 1$. For the **Super-resolution problem** the number of iterations is small and seems to decrease with n both for the two-grid and for the multigrid procedures: more precisely, setting $(x_e)_i = i/N$ the exact solution, in the case of a two-grid method we have 14, 12, 11, 10 and 8 iterations for $N = 32, 64, 128, 256, 512$ and, in the case of a full multigrid method, 14, 13, 13, 12 and 10 iterations for $N = 32, 64, 128, 256, 512$.

5.3. The image restoration of a satellite

We consider the restoration of $\mathcal{S}$ from $\mathcal{S}_q$ in the case of Dirichlet boundary conditions. According to (12), the Point Spread Function is determined by $q = 3$ and $\psi(x_1, x_2) = [4 + 2\cos(x_1) + 2\cos(x_2)]^3 + 1$ so that the resulting generating function is nonnegative and has a unique zero at (π, π) of order 6. Therefore, the associated Toeplitz sequence is asymptotically very ill-conditioned ($\sim [N(n)]^3$) and, despite this bad spectral behavior, the proposed multigrid method is optimal as emphasized by the linear convergence reported in Table 7. We stress that we are applying the ordinary conjugate gradient method as intermediate iteration and the Gauss Seidel

Table 5. Twogrid - $2D$ case: $S_n(f)x = b$, $n = (n_1, n_2)$, $f(\pi,\pi) = 0$, A) $(x_e)_i = i/N$, B) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1$, C) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + 10^{-2}(-)^i$, D) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + 10^{-1}(-)^i$, E) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + (-)^i$,

$q = 1$ $\psi(x_1, x_2) \equiv 1$					
$N = n_1 n_2$	A	B	C	D	E
32^2	5	7	7	7	7
64^2	5	7	7	7	7
128^2	5	7	7	7	7

Table 6. Multigrid - $2D$ case: $S_n(f)x = b$, $n = (n_1, n_2)$, $f(\pi,\pi) = 0$, A) $(x_e)_i = i/N$, B) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1$, C) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + 10^{-2}(-)^i$, D) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + 10^{-1}(-)^i$, E) $(x_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + (-)^i$,

$q = 1$ $\psi(x_1, x_2) \equiv 1$					
$N = n_1 n_2$	A	B	C	D	E
16^2	1	1	1	1	1
32^2	5	7	7	7	7
64^2	5	7	7	7	7
128^2	4	6	6	6	6
256^2	4	6	6	6	6
512^2	4	6	6	6	6

method as smoothing iteration.

Finally, we remark that in the case of noise the regularized systems with $\mu > 0$ (μ Tikhonov parameter) have a better conditioning than in the case of $\mu = 0$: therefore, our multigrid procedure, which is optimal for $\mu = 0$, will be robust since the number of iterations will be bounded by a constant independent both of $N(n)$ and of μ.

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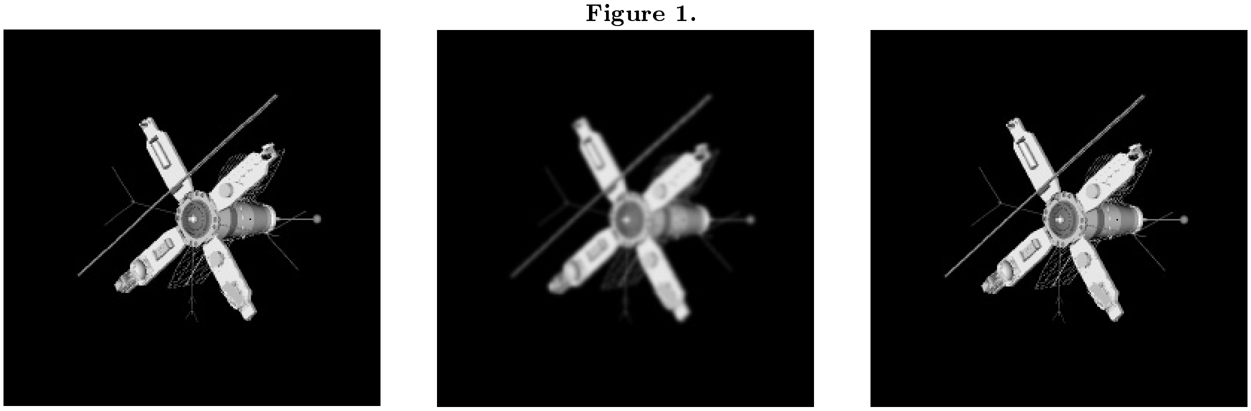
Table 7. Error behavior in $\|\cdot\|_2$ norm.

$\#(Iter.)$	error norm
1	5.471369E+00
5	4.246452E-01
10	3.167303E-02
15	2.537704E-03
20	2.192005E-04
22	8.235485E-05

Figure 1.



True Image (dim: 253×253). Blurred Image Restored Image after 22 iterations.

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