Central Moments, Stochastic Dominance, Moment Rule, and Diversification with an Application

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Abstract:

In this paper, we first develop some properties to state the relationships among central moments, stochastic dominance (SD), risk-seeking stochastic dominance (RSD), and integrals for the general utility functions and the polynomial utility functions of both risk averters and risk seekers. We then introduce the moment rule and establish some necessary and/or sufficient conditions between stochastic dominance and the moment rule for the general utility functions and the polynomial utility functions of both risk averters and risk seekers without imposing the same-location-scale-family condition. Thereafter, we apply the moment rules to develop some properties of portfolio diversification for the general utility functions and the polynomial utility functions for both risk averters and risk seekers. The findings in our paper enable academics and practitioners to draw preferences of both risk averters and risk seekers on their choices of portfolios or assets by using different moments. We illustrate this by using the moment rule tests to compare excess return of 49 industry portfolios from Kenneth French's online data library.

Keywords: Stochastic dominance, central moments, expected-utility maximization, risk aversion, risk seeking, investment behaviors, moment rule

JEL Classification: C00, G11

1 Introduction

Lehmann (1951, 1952, 1955) first compare several sets of distributions that lead to the development of the theory of stochastic dominance (SD). Afterwards, Hadar and Russell (1969, 1971), Hanoch and Levy (1969), and others establish some basic relationships between SD and the preference of different assets to the second order. Extensions of the SD theory to the higher-order include Whitmore (1970), Ekern (1980), Bawa, *et al.* (1985), Muliere and Scarsini (1989), Wong and Li (1999), Li and Wong (1999), Denuit, *et al.* (2013), Post and Koppa (2013), Fang and Post (2017, 2022), Post and Kopa (2017) and many others. Jean (1975) is one of the first few papers that express the moments in terms of successive integrals of a probability density function so that the SD rankings can be compared with moment rankings. Other studies, for example, Brockett and Garven (1998), establish some properties for the relationship between risk, return, skewness, and utility preferences. Studying the high-order moments is important because many studies, for example, Kraus and Litzenberger (1976), Scott and Horvath (1980), Astebro (2003), Cvitanić, Polimenis, and Zapatero (2008), Choi and Nam (2008), Chiu (2010), and Astebro, Mata, and Santos-Pinto (2015) have found that preference of high-order moments plays an important role in asset pricing and many other areas in finance and economics. In this paper, we first extend their work by developing some properties to state the relationships between the n^{th}-order (central) moments, the n^{th}-order SD, the n^{th}-order risk-seeking SD (RSD), and the n^{th}-order [reversed] integrals for both n^{th}- and $(n+1)^{th}$-order [R]SD for general risk-averse [risk-seeking] utility functions and the polynomial utility functions of both risk averters and risk seekers for any order n, including $n = 2, 3$, and 4 as the special cases.

Markowitz (1952a) first introduces the mean-variance (MV) rule for risk averters, and Wong (2007) and others introduce the MV rule for risk seekers because it is well-known that the MV rule for risk averters cannot handle some situations, see, for example, Copeland *et al.* (2005) and Levy (2015) for more information. Studies like Brockett and Garven (1998), and

Meyer, Li, and Rose (2005) prefer other rules than the MV rule. For example, Brockett and Garven (1998) suggest that the SD rule is better than the MV rule. One limitation of the MV rule is that it has not measured any information from high moments, while measuring information from high moments is very important in many empirical data.[1] To circumvent the limitations of the MV rule, in this paper, we extend the MV rule by introducing the moment rules to acquire information from higher moments in the comparison, including the mean-variance-skewness and mean-variance-skewness-kurtosis rules for both risk averters and risk seekers. As far as we know, our paper is the first paper in the literature to introduce the moment rules for both risk averters and risk seekers by establishing some necessary and/or sufficient conditions between SD and the moment rule under some conditions, inferring that the moment rule could be as good as the SD rule under some conditions. In addition, we extend the theory further by removing the same-location-scale-family condition to establish some necessary conditions between SD and the moment rule for both risk averters and risk seekers under some conditions.

Diversification is one of the most important areas in finance. Many studies[2] have developed some properties of diversification related to the MV and SD rules. In this paper, we extend their theories by applying the moment rules to develop some properties of portfolio diversification to compare the preferences of two sets of assets for the general utility functions and the polynomial utility functions of both risk averters and risk seekers. In addition, applying the moment rules to develop some properties of portfolio diversification to compare the preferences between an individual asset, a completely diversified portfolio, and a partially diversified portfolio. We also develop some properties of portfolio diversification

[1]See, for example, Lim (1989), Perez-Quiros and Timmermann (2001), Jondeau and Rockinger (2003, 2006), Cvitanić, Polimenis, and Zapatero (2008), Choi and Nam (2008), Harvey, Liechty, Liechty, and Muller (2010), Grigoletto and Lisi (2011), Buckle, Chen, and Williams (2016), and Do, Brooks, Treepongkaruna, and Wu (2016) for more information.

[2]See, for example, Hadar and Russell (1971), Tesfatsion (1976), Li and Wong (1999), Wong (2007), Egozcue and Wong (2010), Guo and Wong (2016), and Chan, *et al.* (2020) for more information.

that could allow us to compare preferences of some pairs of partially diversified portfolios for any set of independent assets or some sets of dependent assets and the general utility functions of both risk averters and risk seekers. The findings in our paper enable academics and practitioners to draw preferences of both risk averters and risk seekers on their choices of portfolios or assets by using different moments. We illustrate the applications of the moment rules introduced in this paper by using the tests of the moment rules to compare the excess return of 49 industry portfolios from Kenneth French's online data library. We find that the results are reasonably stable from Jan 1992 to Dec 2021. First, around 30% of the portfolios that are dominated by an industry portfolio under a moment rule are also dominated by an industry portfolio under a moment rule from Jan 2002 to Dec 2011. Second, around 50% of the portfolios that are dominated by an industry portfolio under a moment rule from Jan 2002 to Dec 2011 are also dominated by an industry portfolio under a moment rule in the period from Jan 2012 to Dec 2021. Third, around 50% of the portfolios that are dominated by an industry portfolio under a moment rule from Jan 1992 to Dec 2001 are also dominated by an industry portfolio under a moment rule from Jan 2012 to Dec 2021.

The paper is organized as follows. We introduce some definitions and notations in the next section. Section 3 develops some properties on the relationships among central moments, stochastic dominance, and expected utility. Section 4 introduces the moment rule and develops some properties for the moment rule. Section 5 develops some properties of portfolio diversification for the general utility functions and the polynomial utility functions of both risk averters and risk seekers. Section 6 develops some properties for the preferences between some partially-diversified portfolios. Section 7 provides testing procedures for the moment rule. Section 8 illustrates the applicability of the theory developed in our paper by using real-life data. Section 9 concludes our findings.

2 Definitions and notations

Let $\overline{\mathbb{R}}$ be the set of extended real numbers and $\Omega = [a, b]$ be a subset of $\overline{\mathbb{R}}$ in which $a < b$. For random variable $Z = X$ and Y with "*cumulative distribution function*" (CDF) $H = F$ and G and probability density function $h = f$ and g, the mean of Z is defined as $\mu_Z = \mu_H$. The CDF $H_1(x) \equiv H(x) \equiv \mu[a, x]$ of the measure μ is defined on the support $\Omega = [a, b] \subset \overline{\mathbb{R}}$ with $\mu(\Omega) = 1$. For any integer $n > 1$, we define the following:

$$C_H^{(n)} = \int_a^b (x - \mu_H)^n dH(x) \ , \quad H_n(x) = \int_a^x H_{n-1}(t)\, dt \quad , \quad H_n^R(x) = \int_x^b H_{n-1}^R(t)\, dt \ . \qquad (2.1)$$

We note that $H_0(x) = H_0^R(x) = h(x)$, $C_H^{(2)} = \sigma_H^2$ is the variance of H, and $C_H^{(n)}$ is the n^{th}-order central moment for any integer $n \geq 2$. We also note that for any variable Z with CDF H, we will use both $C_Z^{(n)} = C_H^{(n)}$ to be the n^{th}-order central moment of Z for any integer $n \geq 2$. We further let γ_H as the skewness, and κ_H as the kurtosis of H respectively. We call H_n (H_n^R) the n^{th}-order (reversed) integral. We note that H_n is used in the development of the SD theory for risk averters while H_n^R is used in the development of the SD theory for risk seekers, see, for example, Quirk and Saposnik (1962), Hanoch and Levy (1969), Hammond (1974), Levy (2015), and Guo and Wong (2016) and the references therein for more information.

In this paper, we first extend the MV rule[3] for both risk averters and risk seekers introduced by Markowitz (1952a), Wong (2007), and others. It is well-known that the mean-variance rule for risk averters cannot handle some situations, see, for example, the paradox used in the example in p66-67 of Copeland, *et al.* (2005). We modify the example as follows:

Example 2.1 *For any pair of two prospects, X and Y, with probability functions, P_X and P_Y, respectively, such that $P_X(z) = 0.2$ for $z = 3, 5, 7, 9,$ and 11, and $P_Y(z) = 0.2$ for $z = 3, 4, 5, 6,$ and 7.*

One could easily find that X and Y as stated in Example 2.1 do not dominate each other

[3] One may refer to Definition 8 in Wong (2007) for the rule.

by using the MV rule for risk averters. However, it is obvious that X is preferred to Y. To solve the problem, one could apply the MV rule for risk seekers introduced by Wong (2007) and others. Now, when one applies the mean-variance rule for risk seekers, one will conclude that X dominates Y, and thus, conclude that risk seekers will prefer X to Y. However, there are still some limitations of using the mean-variance rules for both risk averters and risk seekers because by using the rules, one could only conclude that risk seekers will prefer X to Y but cannot conclude that risk averters will also prefer X to Y in the paradox as stated in Copeland, *et al.* (2005) and Example 2.1 but, in fact, it is well-known that both risk averters and risk seekers will prefer X to Y in this example. To circumvent the limitation, a well-known solution is to apply the theory of stochastic dominance (SD). To do so, Levy (2015), Guo and Wong (2016), Bai, *et al.* (2021), and others define the n^{th}-order SD. We state the rule briefly here. One may refer to Guo and Wong (2016) for the full definition of the rules.

Definition 2.1 *For any integer n and for any pair of random variables, X and Y, with CDFs, F and G, respectively,*

1. *X is said to dominate Y by the n^{th}-order stochastic dominance for risk averters for $n \geq 1$, denoted by $X \succeq_n Y$, or $F \succeq_n G$ if and only if $F_n(x) \leq G_n(x)$ for each x and $F_k(b) \leq G_k(b)$ for $k = 1, \cdots, n-1$ if $n > 1$, and*

2. *X is said to dominate Y by the n^{th}-order risk-seeking stochastic dominance (RSD)*[4] *for $n \geq 1$, denoted by $X \succeq_n^R Y$ or by $F \succeq_n^R G$, if and only if $F_n^R(x) \geq G_n^R(x)$ for each x in $[a, b]$, and $F_k^R(b) \geq G_k^R(b) \quad$ for $k = 1, \cdots, n-1$ if $n > 1$.*

Applying Definition 2.1 to the paradox as stated in Copeland, *et al.* (2005) and Example 2.1, one will conclude that both risk averters and risk seekers will prefer X to Y and, in

[4] We note that Levy (2015) and others call it RSSD while we follow Guo and Wong (2016) and others to call it RSD.

fact, conclude that all investors with increasing utility, including both risk averters and risk seekers, will prefer X to Y because we have both $X \succeq_1 Y$ and $X \succeq_1^R Y$.

Stochastic dominance is useful for ranking prospects with uncertainty because ranking prospects is equivalent to maximizing the expected utility preferences.[5] To show the advantages of applying SD, we first define utility functions for risk averters and risk seekers (Levy, 2015; Guo and Wong, 2016; Bai, *et al.,* 2020) as the following:

Definition 2.2 *Sets of utility functions,* U_n *and* U_n^R *, for risk averters and risk seekers are:*

$$U_n = \{u : (-1)^i u^{(i)} \leq \ 0\ ,\ i = 1, \cdots, n\} \quad \text{and} \quad U_n^R = \{u : u^{(i)} \geq \ 0\ ,\ i = 1, \cdots, n\}, \qquad (2.2)$$

respective, where $u^{(i)}$ *is the* i^{th} *derivative of the utility function* u*.*

In addition, in this paper, we will develop a theory related to the n^{th}-order polynomial utility function. For this purpose, we define the n^{th}-order polynomial utility function as the following:

Definition 2.3 *If* $u \in U_n$ *or* U_n^R *and* $u^{(n)}$ *is a nonzero constant, then* $u \in U_{np}$ *or* U_{np}^R *is a* n^{th}*-order polynomial utility function for risk averters and risk seekers, respectively.*

We note that it is easy to extend the theory to include non-differentiable utilities.[6] For any investor with $u \in U_n$, it is well known that a negative second derivative for the utility function infers that investors are risk-averse and a positive third derivative for the utility function is a necessary, but not sufficient condition for decreasing absolute risk aversion (DARA).

So far, it is well known that the SD rule is more superior to the MV rule. Thus, this paper aims to improve the MV rule. To do so, we extend the MV rule to be the moment

[5]We follow von Neumann-Morgenstern (1944) to compare the preference among prospects. For an empirical test on SD see Post(2003)

[6]See Wong and Ma (2008).

rule and we will study the relationships between the n^{th}-order central moments with the n^{th}-order (reversed) integrals, the n^{th}-order (risk-seeking) SD, and the moment rule. We discuss the theory in the next section.

3 Theory

In this section, we develop some properties on the relationships among central moments, stochastic dominance, and expected utility. These properties have implications for the moment rule in Section 4 and portfolio diversification in Section 5. We first develop some properties for general utility functions defined in Definition 2.2 and develop properties for polynomial utility functions thereafter in this section.

3.1 General utility functions

We first develop some properties on the relationships among central moments, stochastic dominance, and expected utility for general utility functions defined in Definition 2.2. Since

$$\begin{aligned} \mu_H &= \int_a^b t dH(t) = tH(t)\Big|_a^b - \int_a^b H(t)dt \\ &= b - H_2(b) = a + H_2^R(a) \ , \end{aligned} \tag{3.1}$$

where $H = F$ or G, we have

$$\mu_F - \mu_G = G_2(b) - F_2(b) = F_2^R(a) - G_2^R(a). \tag{3.2}$$

Chan, *et al.* (2020) extend the work by Jean (1975) and others by establishing the following result:

$$G_3(b) - F_3(b) = G_3^R(a) - F_3^R(a) = \frac{1}{2}\left(\sigma_G^2 - \sigma_F^2\right) \ , \tag{3.3}$$

given $\mu_F = \mu_G$.

In this paper, we first extend their results by establishing the following theorem to examine the relationship between the n^{th}-order (central) moments and the n^{th}-order integrals for both n^{th}- and $(n+1)^{th}$-order SD:

Theorem 3.1 *Let $C_H^{(k)}$ be the k^{th}-order central moment for any integer $k \geq 2$. For any given $n \geq 2$, if $C_F^{(k)} = C_G^{(k)}$ for all $2 \leq k < n$ and $\mu_F = \mu_G$, the following statements are equivalent:*

1.
$$G_{n+1}(b) - F_{n+1}(b) = \frac{(-1)^n}{n!}\left(C_G^{(n)} - C_F^{(n)}\right) ; \tag{3.4}$$

2. $G_{n+1}(b) \geq F_{n+1}(b)$ *if and only if* $(-1)^n C_G^{(n)} \geq (-1)^n C_F^{(n)}$;

3. *If* $F \succeq_n G$, *then* $(-1)^n C_F^{(n)} < (-1)^n C_G^{(n)}$; *and*

4. *If* $F \succeq_{n+1} G$, *then* $(-1)^n C_F^{(n)} \leq (-1)^n C_G^{(n)}$.

The proof of Theorem 3.1 is shown in the appendix. Since $F \succeq_n G$ implies $F \succeq_{n+1} G$, one may believe that it is not necessary to have Part 4 of Theorem 3.1. We note that this is not true because Part 4 of Theorem 3.1 includes the case in which $F \succeq_n G$ does not hold but we still have both $F \succeq_{n+1} G$ and $(-1)^n C_G^{(n)} \geq (-1)^n C_F^{(n)}$. Part 3 of Theorem 3.1 does not cover this situation. Thus, we still require to have Part 4 in Theorem 3.1. We note that Fishburn (1980) has derived Part 3 of Theorem 3.1. From Theorem 3.1, we can obtain the following corollary to compare the second-order central moments and SD:

Corollary 3.1 *If $\mu_F = \mu_G$, then*

1. $G_3(b) \geq F_3(b)$ *if and only if* $\sigma_F^2 \leq \sigma_G^2$;

2. *if* $F \succeq_2 G$, *then* $\sigma_F^2 < \sigma_G^2$; *and*

3. *if* $F \succeq_3 G$, *then* $\sigma_F^2 \leq \sigma_G^2$,

where σ_H^2 is the variance of H for $H = F$ or G.

We note that Equation (3.3) is a special case of Theorem 3.1. Now, we turn to develop the following corollary to compare the third-order central moments and SD:

Corollary 3.2 *If $\mu_F = \mu_G$ and $\sigma_F^2 = \sigma_G^2$, then*

1. *$G_4(b) \geq F_4(b)$ if and only if $\gamma_G \leq \gamma_F$;*

2. *if $F \succeq_3 G$, then $\gamma_F > \gamma_G$; and*

3. *if $F \succeq_4 G$, then $\gamma_F \geq \gamma_G$,*

where γ_H is the skewness of H for $H = F$ or G.

Part 1 of Corollary 3.2 shows the necessary and sufficient conditions for the magnitude of the skewnesses with that of the fourth-order integrals for any two distributions, while Part 2 of Corollary 3.2 shows the relationship of the skewnesses with the third-order SD of any two distributions under the conditions of both equal mean and equal variance. From Theorem 3.1, we can also obtain the following corollary to show the relationships between the SD and the kurtosises for any two distributions under the conditions of equal mean, equal variance, and equal skewness:

Corollary 3.3 *If $\mu_F = \mu_G$, $\sigma_F^2 = \sigma_G^2$, and $\gamma_G = \gamma_F$, then*

1. *$G_5(b) \geq F_5(b)$ if and only if $\kappa_G \geq \kappa_F$;*

2. *if $F \succeq_4 G$, then $\kappa_G > \kappa_F$; and*

3. *if $F \succeq_5 G$, then $\kappa_G \geq \kappa_F$,*

where κ_H is the kurtosis of H for $H = F$ or G.

Part 1 of Corollary 3.3 shows the necessary and sufficient conditions for the magnitude of the kurtosises with that of the fifth-order integrals for any two distributions, while Part 2 of Corollary 3.2 shows the relationship between the kurtosises and the fourth-order SD for two different distributions under the conditions of equal mean, equal variance, and equal skewness.

We turn to develop the following theorem as a complement of Theorem 3.1 to show the relationship among the n^{th}-order (central) moments, the n^{th}-order reversed integrals, and the n^{th}- and $(n+1)^{th}$-order RSD:

Theorem 3.2 *Let $C_H^{(k)}$ be the k^{th}-order central moment for any integer $k \geq 2$. For any given $n \geq 2$, if $C_F^{(k)} = C_G^{(k)}$ for all $2 \leq k < n$ and $\mu_F = \mu_G$, the following statements are equivalent:*

1.

$$F_{n+1}^R(a) - G_{n+1}^R(a) \;=\; \frac{1}{n!}\big(C_F^{(n)} - C_G^{(n)}\big) \; ; \qquad (3.5)$$

2. *$F_{n+1}^R(a) \geq G_{n+1}^R(a)$ if and only if $C_F^{(n)} \geq C_G^{(n)}$;*

3. *If $F \succeq_n^R G$, then $C_F^{(n)} > C_G^{(n)}$; and*

4. *If $F \succeq_{n+1}^R G$, then $C_F^{(n)} \geq C_G^{(n)}$.*

The proof of Theorem 3.2 is shown in the appendix. Similar to Theorem 3.1, since $F \succeq_n^R G$ implies $F \succeq_{n+1}^R G$, one may believe that it is not necessary to have Part 4 of Theorem 3.2. We note that this is not true because Part 4 of Theorem 3.2 includes the case in which $F \succeq_n^R G$ does not hold but we still have both $F \succeq_{n+1}^R G$ and $(-1)^n C_G^{(n)} \geq (-1)^n C_F^{(n)}$. From Theorem 3.2, we obtain the following corollary to compare the second-order central moments with RSD:

Corollary 3.4 *If $\mu_F = \mu_G$, then*

1. *$F_3^R(a) \geq G_3^R(a)$ if and only if $\sigma_F^2 \geq \sigma_G^2$;*

2. *if $F \succeq_2^R G$, then $\sigma_F^2 > \sigma_G^2$;*

3. *if $F \succeq_3^R G$, then $\sigma_F^2 \geq \sigma_G^2$; and*

where σ_H^2 is the skewness of H for $H = F$ or G.

We note that Equation (3.3) is a special case of both Theorems 3.1 and 3.2.

Now, we turn to develop the following corollary to compare the third-order central moments with RSDs:

Corollary 3.5 *If $\mu_F = \mu_G$ and $\sigma_F^2 = \sigma_G^2$, then we have*

1. *$F_4^R(a) \geq G_4^R(a)$ if and only if $\gamma_F \geq \gamma_G$;*

2. *if $F \succeq_3^R G$, then $\gamma_F > \gamma_G$; and*

3. *if $F \succeq_4^R G$, then $\gamma_F \geq \gamma_G$,*

where γ_H is the skewness of H for $H = F$ or G.

Part 1 of Corollary 3.5 shows the necessary and sufficient conditions on the magnitude of the skewnesses with the fourth-order reverse integrals for any two distributions while Part 2 of Corollary 3.5 shows the relationship of the skewnesses with the third-order RSD for any two distributions under the conditions of equal mean and equal variance.

From Theorem 3.2, we can also obtain the following corollary to show the relationships between the kurtosises and RSDs for any two distributions under the conditions of equal means, equal variances, and equal skewnesses:

Corollary 3.6 *If $\mu_F = \mu_G$, $\sigma_F^2 = \sigma_G^2$, and $\gamma_G = \gamma_F$, then we have*

1. *$F_5^R(a) \geq G_5^R(a)$ if and only if $\kappa_F \geq \kappa_G$;*

2. *if $F \succeq_4^R G$, then $\kappa_F > \kappa_G$; and*

3. *if $F \succeq_5^R G$, then $\kappa_F \geq \kappa_G$;*

where κ_H is the kurtosis of H for $H = F$ or G.

Part 1 of Corollary 3.5 shows the necessary and sufficient conditions on the magnitude of the kurtosises for any two distributions with the fifth-order reverse integrals while Part 2 of Corollary 3.5 shows the relationship of the kurtosises for two different distributions with the fourth-order RSD under the conditions of equal mean, variance, and skewness.

Guo *et al.* (2014) and others comment that there is no equivalence relationship between moments and SD. In Section 3.1, we find the sufficient condition but not the necessary condition from SD to moment. Is it possible to get the necessary condition, and thus, obtain the equivalence relationship between moments and SD? We explore the answer in the next subsection.

3.2 Polynomial utility functions

We turn to develop some properties on relationships among central moments, stochastic dominance, and expected utility for the polynomial utility functions defined in Definition 2.3. We first develop the following theorem as a complement of Theorem 3.1 to establish some relationships between the n^{th}-order (central) moments and n^{th}-order SD under n^{th}-order polynomial utility functions.

Theorem 3.3 *Let $C_H^{(k)}$ be the k^{th}-order central moment for any integer $k \geq 2$. For any given $n \geq 2$, if $C_F^{(k)} = C_G^{(k)}$ for all $2 \leq k < n$, and $\mu_F = \mu_G$, then for all n^{th}-order polynomial utility function $u \in U_{np}$, the following statements are equivalent:*

1. $F \succeq_n G$,
2. $Eu(F) \geq Eu(G)$, *and*
3. $(-1)^n C_F^{(n)} \leq (-1)^n C_G^{(n)}$.

By applying Theorem 3.1, one could conclude that $G_{n+1}(b) \geq F_{n+1}(b)$ is equivalent to $(-1)^n C_G^{(n)} \geq (-1)^n C_F^{(n)}$ under the assumption of the theorem. Thus, we just need to prove that $G_{n+1}(b) \geq F_{n+1}(b)$ is equivalent to $Eu(F) \geq Eu(G)$ under the assumption of the theorem. The proof of Theorem 3.3 is shown in appendix. Thereafter, by using both Theorem 3.3 and Equation (3.2), we can obtain the following corollaries:

Corollary 3.7 *Suppose $\mu_F = \mu_G$. For any quadratic utility function $u \in U_{2p}$, the following statements are equivalent*

1. $F \succeq_2 G$,
2. $Eu(F) \geq Eu(G)$, *and*
3. $\sigma_F^2 \leq \sigma_G^2$.

We now construct a simple counterexample to show that if the utility function u is cubic, even $\mu_F = \mu_G = \mu$ and $\sigma_F^2 \leq \sigma_G^2$, we may still have $Eu(F) < Eu(G)$.

Example 3.1 *For a cubic utility function, we have:*

$$u(x) = u(\mu) + u'(\mu)(x-\mu) + \frac{u''(\mu)}{2}(x-\mu)^2 + \frac{u'''(\mu)}{6}(x-\mu)^3.$$

As a result, we have:

$$Eu(F) = u(\mu) + \frac{u''(\mu)}{2}\sigma_F^2 + \frac{u'''(\mu)}{6}C_F^{(3)}.$$

Then, we get:

$$Eu(F) - Eu(G) = \frac{u''(\mu)}{2}(\sigma_F^2 - \sigma_G^2) + \frac{u'''(\mu)}{6}(C_F^{(3)} - C_G^{(3)}).$$

This implies that the sign of $Eu(F) - Eu(G)$ depends on the sign of $u''(\mu), u'''(\mu)$, $\sigma_F^2 - \sigma_G^2$, and $C_F^{(3)} - C_G^{(3)}$. Although under the assumed condition, the first term $\frac{u''(\mu)}{2}(\sigma_F^2 - \sigma_G^2)$ is positive, the second term may be negative and this would finally lead the whole term $Eu(F) - Eu(G)$ be negative or positive. For example, consider $u''(\mu) = -2, u'''(\mu) = 6, \sigma_F^2 = 1, \sigma_G^2 = 2, C_F^{(3)} = 1, C_G^{(3)} = 2.5$, we get $Eu(F) - Eu(G) = -0.5 < 0$.

From Theorem 3.3, we obtain the following corollary for any cubic utility function:

Corollary 3.8 *Suppose* $\mu_F = \mu_G$ *and* $\sigma_F^2 = \sigma_G^2$*. For any cubic utility function* $u \in U_{3p}$*, the following statements are equivalent*

1. $F \succeq_3 G$,
2. $Eu(F) \geq Eu(G)$, *and*
3. $\gamma_F \geq \gamma_G$.

From Theorem 3.3, we obtain the following corollary for any quartic utility function:

Corollary 3.9 *Suppose* $\mu_F = \mu_G$, $\sigma_F^2 = \sigma_G^2$, *and* $\gamma_G = \gamma_F$*. For any quartic utility function* $u \in U_{4p}$*, the following statements are equivalent*

1. $F \succeq_4 G$,
2. $Eu(F) \geq Eu(G)$, *and*
3. $\kappa_F \leq \kappa_G$.

Corollary 3.7 tells us that if the means are equal, then under the quadratic utility functions, the preference of the second-order SD is equivalent to the preference (smaller) of the variance (for risk averters); Corollary 3.8 tells us that if the means and variance are equal, then under the cubic utility functions, the preference of the third-order SD is equivalent to the preference (bigger) of the skewness (for risk averters); Corollary 3.9 tells us that if the means, variance, and skewness are equal, then under the quartic utility functions, the preference of the fourth-order SD is equivalent to the preference (smaller) of the kurtosis (for risk averters). In general, Theorem 3.3 tells us that under the conditions that all moments less than n are equal for two assets and under the n^{th}-order polynomial utility functions, then the preference of the n^{th}-order SD is equivalent to the preference of the n^{th}-order central moments (for risk averters). We note that this is very strong result. So far, in literature, for example, Guo *et al.* (2014) comment that there is no equivalence relationship between moments and SD and we believe that our paper is the first paper finds that under some conditions (that is, all moments less than n are equal for two assets and under polynomial utility functions), then the preference of the n^{th}-order SD is equivalent to the preference of the n^{th}-order central moments (for risk averters). In this paper, we hypothesize that this result only holds for polynomial utility functions, not any other nontrivial utility functions. Now, we turn to develop the results for convex utility functions. We first state the following theorem:

Theorem 3.4 *Let $C_H^{(k)}$ be the k^{th}-order central moment for any integer $k \geq 2$. For any given $n \geq 2$, if $C_F^{(k)} = C_G^{(k)}$ for all $2 \leq k < n$, and $\mu_F = \mu_G$. Then, for all n^{th}-order polynomial utility function $u \in U_{np}^R$, the following statements are equivalent:*

1. $F \succeq_n^R G$,
2. $Eu(F) \geq Eu(G)$, *and*
3. $C_F^{(n)} \geq C_G^{(n)}$.

The proof of Theorem 3.4 is shown in appendix. Combining Equation (3.2) and Theorem 3.4, we can obtain the following corollaries. We first obtain the following corollary for any quadratic utility function:

Corollary 3.10 *Suppose $\mu_F = \mu_G$. For any quadratic utility function $u \in U_{2p}^R$, the following statements are equivalent*

1. $F \succeq_2^R G$,
2. $Eu(F) \geq Eu(G)$, *and*
3. $\sigma_F^2 \geq \sigma_G^2$.

We then obtain the following corollary for any cubic utility function:

Corollary 3.11 *Suppose $\mu_F = \mu_G$ and $\sigma_F^2 = \sigma_G^2$. For any cubic utility function $u \in U_{3p}^R$, the following statements are equivalent:*

1. *$F \succeq_3^R G$,*

2. *$Eu(F) \geq Eu(G)$, and*

3. *$\gamma_F \geq \gamma_G$.*

And obtain the following corollary for any quartic utility function:

Corollary 3.12 *Suppose $\mu_F = \mu_G$, $\sigma_F^2 = \sigma_G^2$, and $\gamma_G = \gamma_F$. For any quartic utility function $u \in U_{4p}^R$, the following statements are equivalent:*

1. *$F \succeq_4^R G$,*

2. *$Eu(F) \geq Eu(G)$, and*

3. *$\kappa_F \geq \kappa_G$.*

4 Moment rule

There are many applications of the theory developed in Section 3. In this section, we discuss the applications on the extension of the mean-variance (MV) rule and we call it the moment rule. We also develop some properties for the moment rule in this section.

We first modify the MV rule (for risk averters) as follows:

Definition 4.1 *For any two prospects X and Y with means μ_X and μ_Y and standard deviations σ_X and σ_Y, respectively, X is said to dominate Y by the MV rule for risk averters, denoted by X MV_{RA} Y, if $\mu_X \geq \mu_Y$ and $\sigma_X \leq \sigma_Y$, in which the inequality holds in at least one of the two.*

It is well-known that the mean-variance rule cannot handle some situations. For example, the paradox as shown in Example in p66-67 of Copeland *et al.* (2005). There are several solutions to the paradox.[7] Among them, Wong (2007) introduces the MV rule for risk seekers to solve the paradox. We modify it as follows:

Definition 4.2 *For any two prospects X and Y with means μ_X and μ_Y and standard deviations σ_X and σ_Y, respectively, X is said to dominate Y by the MV rule for risk seekers, denoted by $X \ MV_{RS} \ Y$, if $\mu_X \geq \mu_Y$ and $\sigma_X \geq \sigma_Y$, in which the inequality holds in at least one of the two.*

We note that Meyer, Li, and Rose (2005) use stochastic dominance to examine whether adding internationally based assets to a wholly domestic portfolio generates diversification benefits for an investor. They conclude that stochastic dominance is superior to MV rule. To circumvent the limitation of the MV rule, we first extend the MV rule for risk averters introduced by Markowitz (1952a) and the MV rule for risk seekers introduced by Wong (2007) and others to obtain the following mean-variance-skewness rule for both risk averters and risk seekers to check their preferences on assets based on the first three moments of the distributions:

Definition 4.3 *For any two prospects X and Y with means μ_X and μ_Y, standard deviations σ_X and σ_Y, and skewnesses γ_X and γ_Y, respectively,*

1. *X is said to dominate Y by the mean-variance-skewness rule for risk averters, denoted by $X \ MVS_{RA} \ Y$, if $\mu_X \geq \mu_Y$, $\sigma_X \leq \sigma_Y$, and $\gamma_X \geq \gamma_Y$, and*

2. *X is said to dominate Y by the mean-variance-skewness rule for risk seekers, denoted by $X \ MVS_{RS} \ Y$, if $\mu_X \geq \mu_Y$, $\sigma_X \geq \sigma_Y$, and $\gamma_X \geq \gamma_Y$,*

in which the inequality holds in at least one of the three.

Brockett and Garven (1998) comment that it is possible to have X and Y with positive and equal means, X having a larger variance and lower positive skewness than Y, and yet X has

[7] See Levy (2015) for more information.

a larger expected utility than Y, implying skewness preference for risk averters. Definition 4.3 and the corresponding results in this section could be used to address some concerns for Brockett and Garven (1998). In this paper, we further extend the rule to the following mean-variance-skewness-kurtosis rule for both risk averters and risk seekers:

Definition 4.4 *For any two prospects X and Y with means μ_X and μ_Y, standard deviations σ_X and σ_Y, skewnesses γ_X and γ_Y, and kurtosises, κ_X and κ_Y, respectively,*

1. *X is said to dominate Y by the mean-variance-skewness-kurtosis rule for risk averters, denoted by $X\ MVSK_{RA}\ Y$, if $\mu_X \geq \mu_Y$, $\sigma_X \leq \sigma_Y$, $\gamma_X \geq \gamma_Y$, and $\kappa_X \leq \kappa_Y$,*

2. *X is said to dominate Y by the mean-variance-skewness-kurtosis rule for risk seekers, denoted by $X\ MVSK_{RS}\ Y$, if $\mu_X \geq \mu_Y$, $\sigma_X \geq \sigma_Y$, $\gamma_X \geq \gamma_Y$, and $\kappa_X \geq \kappa_Y$,*

in which the inequality holds in at least one of the four.

We can also extend the rule further to the following mean-variance-skewness-kurtosis-$\cdots$-n^{th}-order central moment rule (we call it first n^{th}-order moments rule or, in short, n-moment rule, or just moment rule) for both risk averters and risk seekers for their preferences on assets based on the first n^{th}-order moments of the distributions:

Definition 4.5 *For any two prospects X and Y with means μ_X and μ_Y, and the k^{th}-order central moments $C_X^{(k)}$ and $C_Y^{(k)}$, respectively, for any $2 \leq k \leq n$,*

1. *X is said to dominate Y by the n-moment rule for risk averters, denoted by $X\ M_{RA}^n\ Y$, if $\mu_X \geq \mu_Y$, and $(-1)^k C_X^{(k)} \leq (-1)^k C_Y^{(k)}$ for any $2 \leq k \leq n$, and*

2. *X is said to dominate Y by the n-moment rule for risk seekers, denoted by $X\ M_{RS}^n\ Y$, if $\mu_X \geq \mu_Y$, and $C_X^{(k)} \geq C_Y^{(k)}$ for any $2 \leq k \leq n$,*

in which the inequality holds in at least one of the above.

Consider the following conjecture for the preferences of both risk averters and risk seekers for the n-moment rule:

Conjecture 1 *For any two prospects X and Y with means μ_X and μ_Y and the k^{th}-order central moments $C_X^{(k)}$ and $C_Y^{(k)}$, respectively, for any $2 \le k \le n$, under some conditions we have*

1. *if $X\ M_{RA}^n\ Y$, then $E\left[u(X)\right] \ge E\left[u(Y)\right]$ for any risk-averse investor with the utility function $u \in U_n$, and*

2. *if $X\ M_{RS}^n\ Y$, then $E\left[u(X)\right] \ge E\left[u(Y)\right]$ for any risk-seeking investor with the utility function $u \in U_n^R$.*

One could easily set an example that Conjecture 1 does not hold if we do not impose any condition. Readers may refer to Chapter 3.13 in Levy (2015) for such a counterexample. In fact, Levy (2015) has shown that $X\ M_{RA}^2\ Y$ is neither sufficient nor necessary for the SSD relationship. Could Conjecture 1 hold true under some conditions? These rules are useful because Wong (2006) and Wong (2007) establish the results[8] to get the necessary conditions between stochastic dominance and the mean-variance rules for risk averters and risk seekers.

Theorem 5 in Wong (2007) tells us that if both X and Y belong to the same location-scale family or the same linear combination of location-scale families, then Conjecture 1 holds for risk averters ($n = 2$) and for risk seekers ($n = 2$). However, the condition that both X and Y belong to the same location-scale family or the same linear combination of location-scale families is very strong. Not many real-life data follow the same-location-scale-family condition. Could we relax in this condition?

Readers may believe that one could use the result from Theorems 3.1 and 3.2 to remove the same-location-scale-family condition. That is true in the sense that they can be used to obtain the necessary but not sufficient conditions for Conjecture 1 as stated in the following theorem:

Theorem 4.1 *For any two prospects X and Y with means μ_X and μ_Y and the k^{th}-order*

[8] Readers may refer to Theorem 5 in Wong (2007).

central moments $C_X^{(k)}$ *and* $C_Y^{(k)}$*, respectively, for any* $2 \le k \le n$*, if* $\mu_X = \mu_Y$ *and if* $C_X^{(k)} = C_Y^{(k)}$ *for any* $2 \le k \le n-1$*, then we have*

1. *if* $E\left[u(X)\right] \ge E\left[u(Y)\right]$ *for any risk-averse investor with the utility function* $u \in U_n$*, then* $X\ M_{RA}^n\ Y$*, and*

2. *if* $E\left[u(X)\right] \ge E\left[u(Y)\right]$ *for any risk-seeking investor with the utility function* $u \in U_n^R$*, then* $X\ M_{RS}^n\ Y$*.*

We note that applying the result from Theorems 3.1 and 3.2 could help us to obtain the necessary but not sufficient conditions for Conjecture 1. On the other hand, employing the result from Theorems 3.3 and 3.4 could get us both necessary and sufficient conditions for Conjecture 1 as the following:

Theorem 4.2 *For any two prospects* X *and* Y *with means* μ_X *and* μ_Y *and the* k^{th}*-order central moments* $C_X^{(k)}$ *and* $C_Y^{(k)}$*, respectively, for any* $2 \le k \le n$*, if* $\mu_X = \mu_Y$ *and if* $C_X^{(k)} = C_Y^{(k)}$ *for any* $2 \le k \le n-1$*, then we have*

1. $X\ M_{RA}^n\ Y$ *if and only if* $E\left[u(X)\right] \ge E\left[u(Y)\right]$ *for any risk-averse investor with the utility function* $u \in U_{np}$*, and*

2. $X\ M_{RS}^n\ Y$ *if and only if* $E\left[u(X)\right] \ge E\left[u(Y)\right]$ *for any risk-seeking investor with the utility function* $u \in U_{np}^R$*.*

Applying the results from Theorems 3.3 and 3.4 does getting us a very nice result that both necessary and sufficient conditions for Conjecture 1 are satisfied. Nonetheless, we need to impose another very strong (equal-$n-1$-moments) assumption of first $n-1$ moments being equal that most real-life data do not satisfied. In addition, it may be the case that practitioners are only interested in knowing the sufficient condition of Conjecture 1, but not the necessary condition of Conjecture 1. In view of this, we develop the following theorem to relax both the same-location-scale-family assumption and the equal-$n-1$-moments assumption:

Theorem 4.3 *For any two prospects X and Y with means μ_X and μ_Y, and the k^{th}-order central moments $C_X^{(k)}$ and $C_Y^{(k)}$, respectively, for any $2 \leq k \leq n$, if $\mu_X = \mu_Y$, then*

1. *if $X \ M_{RA}^n \ Y$, then $E[u(X)] \geq E[u(Y)]$ for any risk-averse investor with the utility function $u \in U_{np}$, and*

2. *if $X \ M_{RS}^n \ Y$, then $E[u(X)] \geq E[u(Y)]$ for any risk-seeking investor with the utility function $u \in U_{np}^R$.*

Theorem 4.3 does relax both the strong same-location-scale-family assumption and the equal-$n-1$-moments assumption that most real-life data do not satisfied. However, Theorem 4.3 requires another assumption that $u \in U_{np}$ or U_{np}^R. it does not require assumption on the distribution (except equal mean), but it is restricted to the types of investors that only belong to polynomial utility functions.

Academics and practitioners may ask: is it possible we do not impose the strong same-location-scale-family assumption, the equal-$n-1$-moments assumption, and we do not restrict to the types of investors that only belong to polynomial utility functions? We get the following theorem for this purpose:

Theorem 4.4 *For any two prospects X and Y with means μ_X and μ_Y, and the k^{th}-order central moments $C_X^{(k)}$ and $C_Y^{(k)}$, respectively, for any $2 \leq k \leq n$, if $\mu_X = \mu_Y$ and if the summation of all the terms in Taylor expansion of the utility after the n term is ignorable, then*

1. *if $X \ M_{RA}^n \ Y$, then $E[u(X)] \geq E[u(Y)]$ for any risk-averse investor with the utility function $u \in U_n$, and*

2. *if $X \ M_{RS}^n \ Y$, then $E[u(X)] \geq E[u(Y)]$ for any risk-seeking investor with the utility function $u \in U_n^R$.*

We note that Theorem 4.4 does relax both the strong same-location-scale-family assumption and the equal-$n-1$-moments assumption that most real-life data do not satisfied. In addition, it does not restrict $u \in U_{np}$ or U_{np}^R. But then, there is no free lunch. The price to pay is that we need to impose the assumption that *the summation of all the terms in Taylor expansion of the utility after the n term is ignorable* that makes Theorem 4.4 "ugly".

Could we relax the strong "equal-mean" condition? The answer is "YES" as we have got two as following:

Theorem 4.5 *For any two prospects X and Y with means μ_X and μ_Y ($\mu_X \neq \mu_Y$), and the k^{th}-order central moments $C_X^{(k)}$ and $C_Y^{(k)}$, respectively, for any $k = 2, 3$, for any $n = 2, 3$ we have*

1. *if $X \ M_{RA}^n \ Y$, then $E\left[u(X)\right] \geq E\left[u(Y)\right]$ for any risk-averse investor with the utility function $u \in U_{np}$, and*

2. *if $X \ M_{RS}^n \ Y$, then $E\left[u(X)\right] \geq E\left[u(Y)\right]$ for any risk-seeking investor with the utility function $u \in U_{np}^R$.*

Theorem 4.6 *For any two prospects X and Y with means μ_X and μ_Y ($\mu_X \neq \mu_Y$), and the k^{th}-order central moments $C_X^{(k)}$ and $C_Y^{(k)}$, respectively, for any $2 \leq k \leq n$,*

1. *if $X \ M_{RA}^n \ Y$, and $C_Y^{(k)} \leq 0$ for any odd integer $k \geq 3$, then $E\left[u(X)\right] \geq E\left[u(Y)\right]$ for any risk-averse investor with the utility function $u \in U_{np}$ and*

2. *if $X \ M_{RS}^n \ Y$ and $C_X^{(k)} \geq 0$ for any odd integer $k \geq 3$, then $E\left[u(X)\right] \geq E\left[u(Y)\right]$ for any risk-seeking investor with the utility function $u \in U_{np}^R$.*

We note that in Part 1 of Theorem 4.6, we only require $C_Y^{(k)} \leq 0$ for any odd integer $k \geq 3$, while the sign of $C_X^{(k)}$ is not required. It can be positive as long as $C_X^{(k)} \geq C_Y^{(k)}$ holds. Similarly, in Part 2 of Theorem 4.6, we only require $C_X^{(k)} \geq 0$ for any odd integer $k \geq 3$, the sign of $C_Y^{(k)}$ is not required. It can be negative as long as $C_X^{(k)} \geq C_Y^{(k)}$ holds.

5 Portfolio Diversification

We now extend the theory developed in Section 4 to develop some properties of portfolio diversification for the general utility functions and the polynomial utility functions of both risk averters and risk seekers. We will apply some results in Hadar and Russell (1971), Tesfatsion (1976), Li and Wong (1999), Wong (2007), Guo and Wong (2016), Chan, *et al.* (2020), and others. First, we apply Theorem 5 in Wong (2007) and Theorem 4.1 to develop the following properties of portfolio diversification to compare the preferences of two sets of assets for the general utility functions of both risk averters and risk seekers:

Theorem 5.1 *For any $i = 1, \cdots, m$, let $\{X_i\}$ and $\{Y_i\}$ be two sets of independent variables with means μ_{X_i} and μ_{Y_i} and the k^{th}-order central moments $C^{(k)}_{X_i}$ and $C^{(k)}_{Y_i}$, respectively, for any $2 \leq k \leq n$, if both X_i and Y_i belong to the same location-scale family or the same linear combination of location-scale families, then, for any $2 \leq k \leq n$, if $\mu_{X_i} = \mu_{Y_i}$ and if $C^{(k)}_{X_i} = C^{(k)}_{Y_i}$ for any $2 \leq k \leq n-1$, then we have*

1. *if $E\left[u(X_i)\right] \geq E\left[u(Y_i)\right]$ for any risk-averse investor with the utility function $u \in U_n$, then $\sum_{i=1}^m \alpha_i X_i \ M^n_{RA} \ \sum_{i=1}^m \alpha_i Y_i$, and*

2. *if $E\left[u(X_i)\right] \geq E\left[u(Y_i)\right]$ for any risk-seeking investor with the utility function $u \in U^R_n$, then $\sum_{i=1}^m \alpha_i X_i \ M^n_{RS} \ \sum_{i=1}^m \alpha_i Y_i$,*

for any $\alpha_i \geq 0, i = 1, \cdots, m$.

Next, we apply Theorem 5 in Wong (2007) and Theorem 4.2 to obtain the following theorem to compare the preferences two sets of assets for the polynomial utility functions of both risk averters and risk seekers:

Theorem 5.2 *For any $i = 1, \cdots, m$, let $\{X_i\}$ and $\{Y_i\}$ be two sets of independent variables with means μ_{X_i} and μ_{Y_i} and the k^{th}-order central moments $C^{(k)}_{X_i}$ and $C^{(k)}_{Y_i}$, respectively, for any $2 \leq k \leq n$, if $\mu_{X_i} = \mu_{Y_i}$ and if $C^{(k)}_{X_i} = C^{(k)}_{Y_i}$ for any $2 \leq k \leq n-1$, then we have*

1. $\sum_{i=1}^{m} \alpha_i X_i \ M_{RA}^n \ \sum_{i=1}^{m} \alpha_i Y_i$ *if and only if* $E\left[u(\sum_{i=1}^{m} \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^{m} \alpha_i Y_i)\right]$ *for any risk-averse investor with the utility function* $u \in U_{np}$*, and*

2. $\sum_{i=1}^{m} \alpha_i X_i \ M_{RS}^n \ \sum_{i=1}^{m} \alpha_i Y_i$ *if and only if* $E\left[u(\sum_{i=1}^{m} \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^{m} \alpha_i Y_i)\right]$ *for any risk-seeking investor with the utility function* $u \in U_{np}^R$*,*

for any $\alpha_i \geq 0, i = 1, \cdots, m$.

We then apply Theorem 5 in Wong (2007) and Theorem 4.3 to obtain the following theorem to compare the preferences two sets of assets for the polynomial utility functions of both risk averters and risk seekers:

Theorem 5.3 *For any* $i = 1, \cdots, m$*, let* $\{X_i\}$ *and* $\{Y_i\}$ *be two sets of independent variables with means* μ_{X_i} *and* μ_{Y_i} *and the* k^{th}*-order central moments* $C_{X_i}^{(k)}$ *and* $C_{Y_i}^{(k)}$*, respectively, for any* $2 \leq k \leq n$*, if* $\mu_{X_i} = \mu_{Y_i}$*, then*

1. *if* $\sum_{i=1}^{m} \alpha_i X_i \ M_{RA}^n \ \sum_{i=1}^{m} \alpha_i Y_i$*, then* $E\left[u(\sum_{i=1}^{m} \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^{m} \alpha_i Y_i)\right]$ *for any risk-averse investor with the utility function* $u \in U_{np}$*, and*

2. *if* $\sum_{i=1}^{m} \alpha_i X_i \ M_{RS}^n \ \sum_{i=1}^{m} \alpha_i Y_i$*, then* $E\left[u(\sum_{i=1}^{m} \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^{m} \alpha_i Y_i)\right]$ *for any risk-seeking investor with the utility function* $u \in U_{np}^R$*,*

for any $\alpha_i \geq 0, i = 1, \cdots, m$.

In addition, applying Theorem 5 in Wong (2007) and Theorem 4.4, we obtain the following theorem to compare the preferences two sets of assets for the general utility functions of both risk averters and risk seekers:

Theorem 5.4 *For any* $i = 1, \cdots, m$*, let* $\{X_i\}$ *and* $\{Y_i\}$ *be two sets of independent variables with means* μ_{X_i} *and* μ_{Y_i} *and the* k^{th}*-order central moments* $C_{X_i}^{(k)}$ *and* $C_{Y_i}^{(k)}$*, respectively, for any* $2 \leq k \leq n$*, if* $\mu_{X_i} = \mu_{Y_i}$ *and if the summation of all the terms in Taylor expansion of the utility after the* n *term is ignorable, then*

1. *if* $\sum_{i=1}^m \alpha_i X_i \ M_{RA}^n \ \sum_{i=1}^m \alpha_i Y_i$, *then* $E\left[u(\sum_{i=1}^m \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^m \alpha_i Y_i)\right]$ *for any risk-averse investor with the utility function* $u \in U_n$, *and*

2. *if* $\sum_{i=1}^m \alpha_i X_i \ M_{RS}^n \ \sum_{i=1}^m \alpha_i Y_i$, *then* $E\left[u(\sum_{i=1}^m \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^m \alpha_i Y_i)\right]$ *for any risk-seeking investor with the utility function* $u \in U_n^R$,

for any $\alpha_i \geq 0, i = 1, \cdots, m$.

Moreover, applying Theorem 5 in Wong (2007) and Theorem 4.5, we obtain the following theorem to compare the preferences of two sets of assets for the polynomial utility functions of both risk averters and risk seekers:

Theorem 5.5 *For any* $i = 1, \cdots, m$, *let* $\{X_i\}$ *and* $\{Y_i\}$ *be two sets of independent variables with means* μ_{X_i} *and* μ_{Y_i} *and the* k^{th}*-order central moments* $C_{X_i}^{(k)}$ *and* $C_{Y_i}^{(k)}$, *respectively, for any* $k = 2, 3$, *for any* $n = 2, 3$ *we have*

1. *if* $\sum_{i=1}^m \alpha_i X_i \ M_{RA}^n \ \sum_{i=1}^m \alpha_i Y_i$, *then* $E\left[u(\sum_{i=1}^m \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^m \alpha_i Y_i)\right]$ *for any risk-averse investor with the utility function* $u \in U_{np}$, *and*

2. *if* $\sum_{i=1}^m \alpha_i X_i \ M_{RS}^n \ \sum_{i=1}^m \alpha_i Y_i$, *then* $E\left[u(\sum_{i=1}^m \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^m \alpha_i Y_i)\right]$ *for any risk-seeking investor with the utility function* $u \in U_{np}^R$,

for any $\alpha_i \geq 0, i = 1, \cdots, m$.

Last, we apply Theorem 5 in Wong (2007) and Theorem 4.6 to obtain the following theorem to compare the preferences of two sets of assets for the polynomial utility functions of both risk averters and risk seekers:

Theorem 5.6 *For any* $i = 1, \cdots, m$, *let* $\{X_i\}$ *and* $\{Y_i\}$ *be two sets of independent variables with means* μ_{X_i} *and* μ_{Y_i} *and the* k^{th}*-order central moments* $C_{X_i}^{(k)}$ *and* $C_{Y_i}^{(k)}$, *respectively, for any* $2 \leq k \leq n$,

1. *if* $\sum_{i=1}^{m} \alpha_i X_i \ M_{RA}^{n} \ \sum_{i=1}^{m} \alpha_i Y_i$, *and* $C_{Y_i}^{(k)} \leq 0$ *for any odd integer* $k \geq 3$ *for any* i, *then* $E\left[u(\sum_{i=1}^{m} \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^{m} \alpha_i Y_i)\right]$ *for any risk-averse investor with the utility function* $u \in U_{np}$ *and*

2. *if* $\sum_{i=1}^{m} \alpha_i X_i \ M_{RS}^{n} \ \sum_{i=1}^{m} \alpha_i Y_i$ *and* $C_{X_i}^{(k)} \geq 0$ *for any odd integer* $k \geq 3$ *for any* i, *then* $E\left[u(\sum_{i=1}^{m} \alpha_i X_i)\right] \geq E\left[u(\sum_{i=1}^{m} \alpha_i Y_i)\right]$ *for any risk-seeking investor with the utility function* $u \in U_{np}^{R}$.

Theorems 5.1 to 5.6 establishe some necessary and sufficient conditions between the preferences of portfolio diversification by using the moment rules and by using expected utility for both risk averters and risk seekers. We note that Theorems 5.1 to 5.6 compare two sets of assets with the same non-negative weight, α_i, in the corresponding i^{th} assets. We turn to establish some necessary and sufficient conditions between the preferences of portfolio diversification by using the moment rules to compare two sets of assets with different weights in the corresponding assets. To do so, we first define

$$\Lambda_n^0 = \left\{ (\lambda_1, \lambda_2, \cdots, \lambda_n)' \in \mathbb{R}^n : \ 0 \leq \lambda_i \leq 1 \text{ for any } i \ , \ \sum_{i=1}^{n} \lambda_i = 1 \right\} \tag{5.1}$$

and call X_i be an *an individual asset*, $\frac{1}{n}\sum_{i=1}^{n} X_i$ be the *completely diversified portfolio*, and $\sum_{i=1}^{n} \lambda_i X_i$ be a *partially diversified portfolio* if there exists i such that $0 < \lambda_i < 1$ and $\lambda_i \in \Lambda_n^0$. We then extend Theorem 12 in Li and Wong (1999) to obtain the following theorems to compare preference among an individual asset, a completely diversified portfolio, and a partially diversified portfolio for the general utility functions of both risk averters and risk seekers:

Theorem 5.7 *For any* $X_1, \cdots, X_n$ *with* $n \geq 2$, *if* $X_1, \cdots, X_n$ *are i.i.d., then we have*

1. $\frac{1}{n}\sum_{i=1}^{n} X_i \ M_{RA}^{2} \ \sum_{i=1}^{n} \lambda_i X_i \ M_{RA}^{2} \ X_i$ *and* $E\left[u\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \geq E\left[u\left(\sum_{i=1}^{n} \lambda_i X_i\right)\right] \geq E\left[u\left(\ X_i\right)\right]$ *for any risk-averse investor with the utility function* $u \in U_2$, *and*

2. $X_i \ M_{RS}^{2} \sum_{i=1}^{n} \lambda_i X_i M_{RS}^{2} \frac{1}{n}\sum_{i=1}^{n} X_i$ *and* $E\left[u\left(X_i\right)\right] \geq E\left[u\left(\sum_{i=1}^{n} \lambda_i X_i\right)\right] \geq E\left[u\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right]$ *for any risk-seeking investor with the utility function* $u \in U_2^{R}$, *and*

for any $(\lambda_1, \cdots, \lambda_n) \in \Lambda_n$ *defined in Equation (5.1).*

Theorem 5.7 establishes the preferences among individual asset, partially-diversified portfolio, and completely-diversified portfolio by using both moment rule and expected-utility rule for both risk averters and risk seekers.

6 Majorization

Using the results in Section 5, one can compare the preferences among individual asset, partially-diversified portfolio, and completely-diversified portfolio, but cannot compare the preferences between two partially-diversified portfolios. To circumvent the limitation, we apply the results in Egozcue and Wong (2010) to compare the preferences between some partially-diversified portfolios in this section. To do so, we first define

$$\Lambda_n = \left\{ (\lambda_1, \lambda_2, \cdots, \lambda_n)' \in \mathbb{R}^n : 1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\} . \tag{6.1}$$

We then follow Hardy, *et al.* (1934) and others to make the following definition:

Definition 6.1 *Let* $\vec{a}_n, \vec{b}_n \in \Lambda_n$ *defined in (6.1).* b_n *is defined to majorize* $\vec{a}_n$*, represented by* $\vec{b}_n \succeq_M \vec{a}_n$*, if* $\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} a_i$*, for any* $k = 1, 2, \cdots, n$*.*

Applying Theorem 4.2, Theorem 7 in Egozcue and Wong (2010), and Theorem 3.8 in Guo and Wong (2016), we obtain the following theorem to compare preference of any two partially diversified portfolios for any set of independent assets and for the general utility functions of both risk averters and risk seekers:

Theorem 6.1 *For* $n > 1$*, let* $\vec{a}_n, \vec{b}_n \in \Lambda_n$ *and* $\vec{X}_n = (X_1, \cdots, X_n)'$ *in which* $X_1, \cdots, X_n$ *are i.i.d, if* $\vec{b}_n \succeq_M \vec{a}_n$*, then*

1. $\vec{a}_n' \vec{X}_n \; M_{RA}^2 \; \vec{b}_n' \vec{X}_n$ *and* $E\left[u\left(\vec{a}_n' \vec{X}_n\right)\right] \geq E\left[u\left(\vec{b}_n' \vec{X}_n\right)\right]$ *for any risk-averse investor with the utility function* $u \in U_2$*, and*

2. $\vec{b}_n'\vec{X}_n \ M_{RS}^2 \ \vec{a}_n'\vec{X}_n$ *and* $E\left[u\left(\vec{b}_n'\vec{X}_n\right)\right] \geq E\left[u\left(\vec{a}_n'\vec{X}_n\right)\right]$ *for any risk-seeking investor with the utility function* $u \in U_2^R$.

Can we drop the i.i.d. assumption to compare non-i.i.d. portfolio? Samuelson (1967) argues that, in general, we can't, but he does obtain some results to drop the i.i.d. assumption. To follow Samuelson's idea to drop the i.i.d. assumption, we apply Theorem 6.1, Corollary 9 in Egozcue and Wong (2010), and Corollary 3.2 in Guo and Wong (2016) to obtain the following corollary to compare preferences of some pairs of two partially diversified portfolios for some sets of dependent assets and for the general utility functions of both risk averters and risk seekers:

Corollary 6.1 *For* $n > 1$, $\vec{X}_n = (X_1, \cdots, X_n)'$ *is a series of dependent or independent random variables, For any* $\vec{a}_n, \vec{b}_n \in \Lambda_n$, *if there exist i.i.d.* $\vec{Y}_n = (Y_1, \cdots, Y_n)'$ *and* P_{nn} *in which* $\vec{X}_n = P_{nn}\vec{Y}_n$ *such that* $\vec{b}_n'P_{nn} \succeq_M \vec{a}_n'P_{nn}$ *with* $\vec{a}_n'P_{nn}$, *then*

1. $\vec{a}_n'\vec{X}_n \ M_{RA}^2 \ \vec{b}_n'\vec{X}_n$ *and* $E\left[u\left(\vec{a}_n'\vec{X}_n\right)\right] \geq E\left[u\left(\vec{b}_n'\vec{X}_n\right)\right]$ *for any risk-averse investor with the utility function* $u \in U_2$, *and*

2. $\vec{b}_n'\vec{X}_n \ M_{RS}^2 \ \vec{a}_n'\vec{X}_n$ *and* $E\left[u\left(\vec{b}_n'\vec{X}_n\right)\right] \geq E\left[u\left(\vec{a}_n'\vec{X}_n\right)\right]$ *for any risk-seeking investor with the utility function* $u \in U_2^R$.

Last, we turn to apply Theorem 6.1, Corollary 12 in Egozcue and Wong (2010), and Corollary 3.3 in Guo and Wong (2016) to obtain the following corollary to compare preferences of some pairs of two partially diversified portfolios for some sets of dependent assets and for the general utility functions of both risk averters and risk seekers:

Corollary 6.2

For $n > 1$, $\vec{X}_n = (X_1, \cdots, X_n)'$ *and* $\vec{Y}_n = (Y_1, \cdots, Y_n)'$ *are two series of dependent or independent random variables and* $\vec{V}_n = (V_1, \cdots, V_n)'$ *and* $\vec{W}_n = (W_1, \cdots, W_n)'$ *are two series of i.i.d. random variables. For any* $\vec{a}_n, \vec{b}_n \in \Lambda_n$, *if there exist* P_{nn} *and* Q_{nn} *such that*

$\vec{a}_n' P_{nn}$, $\vec{b}_n' Q_{nn} \in \Lambda_n$ *and* $\vec{b}_n' Q_{nn} \succeq_M \vec{a}_n' P_{nn}$, $\vec{X}_n = P_{nn}\vec{V}_n$, $\vec{Y}_n = Q_{nn}\vec{W}_n$, $V_i \succeq_2 W_i$ *for all* $i = 1, 2, ..., n$; *then*

1. $\vec{a}_n'\vec{X}_n \ M_{RA}^2 \ \vec{b}_n'\vec{Y}_n$ *and* $E\left[u\left(\vec{a}_n'\vec{X}_n\right)\right] \geq E\left[u\left(\vec{b}_n'\vec{Y}_n\right)\right]$ *for any risk-averse investor with the utility function* $u \in U_2$, *and*

2. $\vec{b}_n'\vec{X}_n \ M_{RS}^2 \ \vec{a}_n'\vec{Y}_n$ *and* $E\left[u\left(\vec{b}_n'\vec{X}_n\right)\right] \geq E\left[u\left(\vec{a}_n'\vec{Y}_n\right)\right]$ *for any risk-seeking investor with the utility function* $u \in U_2^R$.

7 Moment rule tests

In this section, we describe the testing procedure to test the performance of assets by using the moment rule up to the fourth order. To do so, we let null hypothesis H_0^1 be $H_0^1 : \mu_X < \mu_Y$, H_0^2 be $H_0^2 : \sigma_X^2 > \sigma_Y^2$, H_{0S}^2 be $H_{0S}^2 : \sigma_X^2 < \sigma_Y^2$, H_0^3 be $H_0^3 : \gamma_X < \gamma_Y$, H_0^4 be $H_0^4 : \kappa_X > \kappa_Y$, H_{0S}^4 be $H_{0S}^4 : \kappa_X < \kappa_Y$. Further, we let H_A^i be alternative hypothesis of H_0^i for $i = 1, 2, 3, 4$ and H_{AS}^i be alternative hypothesis of H_{0S}^i for $i = 2, 4$.

To test whether $X \ MV_{RA} \ Y (X \ MV_{RS} \ Y)$, we simply test the joint null hypotheses $H_0^{MV} : H_0^1 \cup H_0^2 (H_{0S}^{MV} : H_0^1 \cup H_{0S}^2)$, if we reject the joint null hypotheses, we will conclude that $X \ MV_{RA} \ Y (X \ MV_{RS} \ Y)$ because the joint alternative hypotheses are $H_A^{MV} : H_A^1 \cap H_A^2 (H_{AS}^{MV} : H_A^1 \cap H_{AS}^2)$. On the other hand, to test whether $X \ MVS_{RA} \ Y (X \ MVS_{RS} \ Y)$, we simply test the joint null hypotheses $H_0^{MVS} : H_0^1 \cup H_0^2 \cup H_0^3 (H_{0S}^{MV} : H_0^1 \cup H_{0S}^2 \cup H_0^3)$, if we reject the joint null hypotheses, we will conclude that $X \ MVS_{RA} \ Y (X \ MVS_{RS} \ Y)$ because the joint alternative hypotheses are $H_A^{MVS} : H_A^1 \cap H_A^2 \cap H_A^3 (H_{AS}^{MVS} : H_A^1 \cap H_{AS}^2 \cap H_A^3)$. Last, but not the least, to test whether $X \ MVSK_{RA} \ Y (X \ MVSK_{RS} \ Y)$, we simply test the joint null hypotheses $H_0^{MVSK} : H_0^1 \cup H_0^2 \cup H_0^3 \cup H_0^4 (H_{0S}^{MVSK} : H_0^1 \cup H_{0S}^2 \cup H_0^3 \cup H_{0S}^4)$. If we reject the joint null hypotheses, we will conclude that $X \ MVSK_{RA} \ Y (X \ MVSK_{RS} \ Y)$ because the joint alternative hypotheses are $H_A^{MVSK} : H_A^1 \cap H_A^2 \cap H_A^3 \cap H_A^4 (H_{AS}^{MVSK} : H_A^1 \cap H_{AS}^2 \cap H_A^3 \cap H_{AS}^4)$.

Because H_0^{MV}, H_{0S}^{MV}, H_0^{MVS}, H_{0S}^{MVS}, H_0^{MVSK}, or H_{0S}^{MVSK}, can be viewed as a composite of multiple hypotheses, we can test those individual hypotheses separately and see whether all of them are rejected in order to reject H_0^{MV}, H_{0S}^{MV}, H_0^{MVS}, H_{0S}^{MVS}, H_0^{MVSK}, or H_{0S}^{MVSK}. For convenient, we let the p-values of H_0^{MV}, H_{0S}^{MV}, H_0^{MVS}, H_{0S}^{MVS}, H_0^{MVSK}, and H_{0S}^{MVSK} as the largest p-values for their corresponding multiple hypotheses. Take H_0^{MVS} as an example, we can find out the p-values for testing H_0^1, H_0^2, and H_0^3, say p_1, p_2 and p_3, then we reject H_0^{MVS} at α significant level if $p^{MVS} = max\{p_i\} \leq \alpha$ for $i = 1, 2, 3$. In order to test H_0^1, H_0^2, and H_{0S}^2 we use the commonly-used T and F tests. To test H_0^3 b, H_0^4 be and H_{0S}^4, we use the bootstrap method (Efron and LePage, 1992; Shao and Tu, 2012).

Since the existing statistical tests for SD, see, for example, Davidson and Duclos (2000), Barrett and Donald (2003), Post and Versijp (2007), Bai, *et al.* (2011, 2015), and Ng, *et al.* (2017), are generally computationally intensive due to the need to use resampling methods, researchers may not want to use the SD tests. In this situation, they could just apply the moment rule tests. Using the moment tests, readers should be able to draw similar conclusions in Vinod (2004), Tsang, *et al.* (2016). Chan, *et al.* (2020), Lv, *et al.* (2021), and many others when one uses their data.

8 Applications

In this section, we illustrate the applicability of the theory developed in our paper by using real-life data.[9] We use the excess return of 49 industry average value-weighted portfolios from Kenneth French's online data library to illustrate the statistical test on H_0^{MVS}, H_{0S}^{MVS},

[9]There are various studies that try to apply the SD approach in the real data. For example, Qiao, *et al.* (2014) and Clark, *et al.* (2016) find that risk averters prefer investing spot to futures while risk seekers prefer investing futures to spot, Wong, *et al.* (2008) conclude that third-order risk averters prefer investing in some Asian hedge funds to other Asian hedge funds, Chan, *et al.* (2020) find that the third-order risk averters prefer investing in the S&P 500 index to the Nasdaq 100 index and the third-order risk seekers prefer investing in the Nasdaq 100 index to the S&P 500 index. Vinod (2004) examines mutual funds and finds fourth-order SD dominance among the funds. Kallio and Hardoroudi (2019) obtain new results for fourth and fifth-order stochastic dominance. On the other hand, Hoang, *et al.* (2015) find that risk-averse investors prefer not to include gold while risk-seeking investors prefer to include it in their portfolios. Chui, *et al.* (2020) use SD test to check whether the market is efficient.

H_0^{MV}, and H_{0S}^{MV}. The portfolios we used in our analysis contain almost all common stocks listed on NYSE, AMEX, and NASDAQ, which are formed by using the four-digit standard industrial classification code.[10]

We denote the portfolios by their names in Kenneth French's online data library. In addition, we use the one-month US Treasury bill as a risk-free rate to help us calculate the excess returns. In this paper, we mainly focus on the monthly excess return from Jan 1992 to December 2021. All data is pulled directly from Kenneth French's online data library.

We first divide our data into three sub-periods, including 1) Jan 1992 to Dec 2001, 2) Jan 2002 to Dec 2011, and 3) Jan 2012 to Dec 2021. We then compare the portfolios by using the relevant test on H_0^{MVS}, H_{0S}^{MVS}, H_0^{MV}, and H_{0S}^{MV}. For any portfolio that is dominated by any other portfolio in MV_{RA}, MV_{RS}, MVS_{RA}, or MVS_{RS} sense for at least two periods, we will show them in Table 1. In Table 1, the second column represents the minimum p-value among all the 48 p-values for testing hypothesis H_0^{MVS} in which Y is the excess return of the industry portfolio denoted in the first column, and X is the excess return of another industry portfolio. To be more specific, under MVS_{RA}, we will perform 48 times of the test for each Y (in order to compare Y with all the other 48 portfolios) and display the smallest p-value among the tests in the second column when the p-value is smaller than 10%. The same logic is applied to the third, fourth, and fifth columns, which represent the minimum p-values among all the 48 p-values for testing hypothesis H_{0S}^{MVS}, H_0^{MV}, and H_{0S}^{MV}, respectively, in which Y is the excess return of the industry portfolio denoted in the first column and X is the excess return of another industry portfolio when the minimum p-value is smaller than 10%. We also compares all the relevant moments for MV_{RA}, MV_{RS}, MVS_{RA}, or MVS_{RS} by using the corresponding 90% bootstrap confidence interval (CI). We will denote "All" if the confidence interval results are consistent with the test results in all of our hypotheses, "Partial" if the confidence interval results are consistent with one or some but not all the test

[10]See https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html for more details about the portfolios.

results in our hypotheses, and "None" if the confidence interval results are not consistent with any of the test results in all of our hypotheses.

From Table 1, we first discuss the results for the period from Jan 1992 to Dec 2001. The results show that "Other" is dominated by at least one industry portfolio in the sense of MVS_{RS}; that is, there exists an industry portfolio that has a significantly higher return, higher variance, and higher skewness compared to "Other". For the MV_{RA} relationship, we find that "Toys", "Gold", "Boxes", and "Other" are dominated by at least one industry portfolio; that is, there exists an industry portfolio that has a significantly higher return and lower variance compared to "Toys", "Gold", "Boxes", or "Other". For the MV_{RS} relationship, we find that "Food", "Toys", "Util", "PerSv", "Boxes", "Whlsl", and "Other" are dominated by at least one industry portfolio; that is, there exists an industry portfolio has significantly higher return and higher variance compared to "Food", "Toys", "Util", "PerSv", "Boxes", "Whlsl", and "Other". The results from the confidence intervals are all consistent with the results from the hypothesis test.

We then discuss the results for the period from Jan 2002 to Dec 2011. From the table, "Books" and "Other" are found to be dominated by at least one industry portfolio in the sense of MV_{RA}. For the MV_{RS} relationship, all portfolios displayed in Table 1 in the period are dominated by at least one industry portfolio. On the other hand, in the period from Jan 2012 to Dec 2021, "Gold" is dominated by at least one industry portfolio in the sense of MVS_{RA}; that is, there exists an industry portfolio that has a significantly higher return, lower variance, and higher skewness compared to "Gold"; for the MVS_{RS} relationship, "Books", "Util", "Telem", "PerSv", and "Paper" are dominated by at least one industry portfolio; for the MV_{RA} relationship, both "Gold" and "PerSv" are dominated by at least one industry portfolio; and for the MV_{RS} relationship, except "Gold", all portfolios shown in Table 1 in the period are dominated by at least one industry portfolio. The results from the confidence interval are all consistent with those from the hypothesis test.

Table 1: Moment rule tests

Industry	MVS_{RA}	MVS_{RS}	MV_{RA}	MV_{RS}	CI
		Jan 1992 to Dec 2001			
Food	n.a.	n.a.	n.a.	0.0778*	All
Toys	n.a.	n.a.	0.0825*	0.0407**	All
Gold	n.a.	n.a.	0.0912*	n.a.	All
Util	n.a.	n.a.	n.a.	0.0811*	All
PerSv	n.a.	n.a.	n.a.	0.0903*	All
Boxes	n.a.	n.a.	0.0546*	0.0573*	All
Whlsl	n.a.	n.a.	n.a.	0.0625*	All
Other	n.a.	0.0834*	0.0339**	0.0395**	All
		Jan 2002 to Dec 2011			
Food	n.a.	n.a.	n.a.	0.0907*	All
Beer	n.a.	n.a.	n.a.	0.0964*	All
Toys	n.a.	n.a.	n.a.	0.0968*	All
Books	n.a.	n.a.	0.0610	0.0394**	All
Hshld	n.a.	n.a.	n.a.	0.0908*	All
Drugs	n.a.	n.a.	n.a.	0.0395**	All
Telcm	n.a.	n.a.	n.a.	0.0582*	All
PerSv	n.a.	n.a.	n.a.	0.0971*	All
Paper	n.a.	n.a.	n.a.	0.0960*	All
Other	n.a.	n.a.	0.0811*	0.0480**	All
		Jan 2012 to Dec 2021			
Food	n.a.	n.a.	n.a.	0.0194**	All
Beer	n.a.	n.a.	n.a.	0.0568*	All
Books	n.a.	0.0898*	n.a.	0.0898*	Partial
Hshld	n.a.	n.a.	n.a.	0.03808**	All
Drugs	n.a.	n.a.	n.a.	0.0850*	All
Gold	0.0782*	n.a.	0.0689*	n.a.	Partial
Util	n.a.	0.0793*	n.a.	0.0167**	All
Telcm	n.a.	0.0943*	n.a.	0.0339**	All
PerSv	n.a.	0.0567*	0.0650*	0.0567*	Partial
Paper	n.a.	0.0833*	n.a.	0.0297**	All
Boxes	n.a.	n.a.	n.a.	0.0737*	All
Whlsl	n.a.	n.a.	n.a.	0.0740*	None
Other	n.a.	n.a.	n.a.	0.0385**	All

The *, and ** denote the significance at 10%, and 5%, respectively.

In summary from all the results, we notice that around 30% of the portfolios are dominated by at least one industry portfolio in MVS_{RA}, MVS_{RS}, MV_{RA}, or MV_{RS} sense in the period from Jan 1992 to Dec 2001 and are dominated by at least one industry portfolio in MVS_{RA}, MVS_{RS}, MV_{RA}, or MV_{RS} sense during the period from Jan 2002 to Dec 2011. Second, around 50% of the portfolios are dominated by at least one industry portfolio in MVS_{RA}, MVS_{RS}, MV_{RA}, or MV_{RS} sense in the period from Jan 2002 to Dec 2011 and are also dominated by at least one industry portfolio in MVS_{RA}, MVS_{RS}, MV_{RA}, or MV_{RS} sense in the period from Jan 2012 to Dec 2021. Third, around 50% of the portfolios are dominated by at least one industry portfolio in MVS_{RA}, MVS_{RS}, MV_{RA}, or MV_{RS} sense in the period from Jan 1992 to Dec 2001 and are dominated by at least one industry portfolio in MVS_{RA}, MVS_{RS}, MV_{RA}, or MV_{RS} sense in the period from Jan 2012 to Dec 2021. Most of the results from the confidence interval are consistent with those from the hypothesis test, except they only partially support the results for "Books", "Gold", and "PerSv". The result of the confidence interval does not support that of the hypothesis test for "Whlsl".

Kindly noted that the empirical application is not conclusive. Significant amount of industries that is classified as significantly dominated seems not hold over time in this study. One possible interpretation is that the location, dispersion and/or shape of the probability distributions of the industries could change over time. Future research could focus on the evaluation of the statistical properties of the proposed tests using Monte Carlo simulations, and the development of universally valid statistical inference methods based on statistical subsampling and moment estimation methods.

9 Concluding remarks

In this paper, we first extend the work of Markowitz (1952a), Tobin (1958), Chan, *et al.* (2020), and others by developing some theorems to state the relationships among central moments, stochastic dominance (SD), risk-seeking stochastic dominance (RSD), and integrals and establishing the relationship between the n^{th}-order (central) moments and the n^{th}-order [reversed] integrals for both n^{th}- and $(n+1)^{th}$-order [R]SD for general risk-averse [risk-seeking] utility functions and the polynomial utility functions of both risk averters and risk seekers for any order n, including $n = 2, 3$, and 4 as the special cases. We then apply the relationships to extend the mean-variance (MV) rule established by Markowitz (1952a), Wong (2007), and others by introducing the moment rule. As far as we know, our paper is the first introduces it in the literature, including the mean-variance-skewness rule, the mean-variance-skewness-kurtosis rule, for both risk averters and risk seekers.

Wong (2006, 2007) establish the necessary conditions between stochastic dominance and the mean-variance rules for both risk averters and risk seekers when the assets belong to the same location-scale family or the same linear combination of location-scale families. In this paper, we extend the theory further by removing the same-location-scale-family condition to establish some necessary conditions between SD and the moment rule for both risk averters and risk seekers under some conditions. Thereafter, we apply the moment rules to develop some properties of portfolio diversification for the general utility functions and the polynomial utility functions of both risk averters and risk seekers. Last, we incorporate the idea of majorization with the moment rules to develop some properties of portfolio diversification for the general utility functions to compare the preferences between two partially diversified portfolios.

The findings in our paper enable academics and practitioners to draw preferences of both risk averters and risk seekers on their choices of portfolios or assets by using different moments. We illustrate this point by using the moment rule tests to compare the excess return of 49 industry portfolios from Kenneth French's online data library. We find that the results are reasonably stable from Jan 1992 to Dec 2021. First, around 30% of the portfolios that are dominated by an industry portfolio under a moment rule are also dominated by an industry portfolio under a moment rule from Jan 2002 to Dec 2011. Second, around 50% of the portfolios that are dominated by an industry portfolio under a moment rule from Jan 2002 to Dec 2011 are also dominated by an industry portfolio under a moment rule in the period from Jan 2012 to Dec 2021. Third, around 50% of the portfolios that are dominated by an industry portfolio under a moment rule from Jan 1992 to Dec 2001 are also dominated by an industry portfolio under a moment rule from Jan 2012 to Dec 2021. Significant amount of industries that are classified as significantly dominated seems not hold over time in this study. One possible interpretation is that the location, dispersion, and/or shape of the probability distributions of the industries have changed over time. Future research could focus on the evaluation of the statistical properties of the proposed tests using Monte Carlo simulations, and the development of universally valid statistical inference methods based on statistical subsampling and moment estimation methods.

Higher orders stochastic dominance and risk measure are useful in many empirical studies, see, for example, Tehranian (1980) and Ogryczak and Ruszczyński (1999). Meyer, Li, and Rose (2005) use stochastic dominance to examine whether adding internationally-based assets to a wholly domestic portfolio generates diversification benefits for an investor. They conclude that stochastic dominance is superior to the mean-variance rule. Further extensions could examine whether stochastic dominance is superior to the moment rule.

Extensions of our paper could also include developing properties between central moments with prospect and Markowitz stochastic dominance for investors with S-shaped or reverse S-shaped utility function, developing the moment rule, establishing the necessary and/or sufficient conditions between prospect and Markowitz stochastic dominance, and developing some properties of portfolio diversification for investors with S-shaped or reverse S-shaped utility function.[11] Readers may read Levy and Wiener (1998), Levy and Levy (2002, 2004), Post and Levy (2005), and Wong and Chan (2008) for more information on prospect and Markowitz stochastic dominance, read Egozcue, *et al.* (2011) and Ortobelli Lozza, *et al.*

[11]S-shaped utility function is proposed by Kahneman and Tversky (1979) and reverse S-shaped utility function is proposed by Markowitz (1952b)

(2018) for the diversification properties for other types of investors, including investors with S-shaped or reverse S-shaped utility function. In addition, our paper develops some properties by using the information of higher-order moments but has not used any information on the joint effects from moments. Thus, extensions of our paper could study the joint effects of moments, see, for example, Martellini and Ziemann (2009), Kostakis, Muhammad, and Siganos (2012), Lambert and Hubner (2013), Vo and Tran (2020), and many others.

Appendix

Proof of Theorem 3.1.

We prove Part 1 of the theorem by induction. Under the assumption that $\mu_F = \mu_G$, Chan, *et al.* (2020) have already proved formula (3.4) for $n = 2$ (see (3.3)). Now we prove that if the theorem holds for all $2 \leq k \leq n-1$, then it also holds for n. By the induction hypothesis, we have

$$G_{k+1}(b) - F_{k+1}(b) = \frac{(-1)^k}{k!}\big(C_G^{(k)} - C_F^{(k)}\big), \quad \forall 2 \leq k \leq n-1.$$

The assumption of the theorem for n implies that $C_G^{(k)} = C_F^{(k)}$ for all $2 \leq k < n$. Hence we have $G_k(b) = F_k(b)$ for all $2 \leq k \leq n$. We also have $G_1(b) = 1 = F_1(b)$.

To get (3.4) for n, we note that for H where H can be F or G:

$$\begin{aligned}
C_H^{(n)} &= \int_a^b \big(x-\mu_H\big)^n dH(x) \\
&= \big(x-\mu_H\big)^n H(x)\Big|_a^b - n\int_a^b \big(x-\mu_H\big)^{n-1} H(x)dx \\
&= \big(b-\mu_H\big)^n - n\big(x-\mu_H\big)^{n-1} H_2(x)\Big|_a^b + \frac{n!}{(n-2)!}\int_a^b \big(x-\mu_H\big)^{n-2} H_2(x)dx \\
&= \sum_{k=1}^{n} \frac{(-1)^{k+1} n!}{(n+1-k)!}\big(b-\mu_H\big)^{n+1-k} H_k(b) + \frac{n!}{(-1)^n} H_{n+1}(b).
\end{aligned}$$

Hence

$$\begin{aligned}
& C_G^{(n)} - C_F^{(n)} \\
= & \sum_{k=2}^{n} \frac{(-1)^{k+1} n!}{(n+1-k)!}\big(b-\mu\big)^{n+1-k}[G_k(b) - F_k(b)] + \frac{n!}{(-1)^n}[G_{n+1}(b) - F_{n+1}(b)] \\
= & \frac{n!}{(-1)^n}[G_{n+1}(b) - F_{n+1}(b)]
\end{aligned}$$

which is precisely (3.4) for n. Thus, the assertion of Part 1 of Theorem 3.1 holds. Part 2 and part 4 of Theorem 3.1 can be obtained from using the result from Part 1 of Theorem 3.1 holds. We turn to prove Part 3 of Theorem 3.1.

To prove Part 3, we will try to prove $F \succeq_n G$ implies $G_{n+1}(b) > F_{n+1}(b)$ under the assumption of Theorem 3.1. Since $G_n(x) \geq F_n(x)$ for each x and $G_n(x) > F_n(x)$ for at least one x_0 in $[a,b]$ implies $\int_a^b G_n(x)dx > \int_a^b F_n(x)dx$, that is $G_{n+1}(b) > F_{n+1}(b)$. Thus, the Part 3 of Theorem 3.1 holds. $\square$

Proof of Theorem 3.2. We prove Part 1 of the theorem by induction. Under the assumption that $\mu_F = \mu_G$, Chan, *et al.* (2020) have already proved formula (3.5) for $n = 2$ (see (3.3)). Now we prove that if the theorem holds for all $2 \le k \le n-1$, then it also holds for n. By the induction hypothesis, we have

$$F_{k+1}^R(a) - G_{k+1}^R(a) = \frac{1}{k!}\big(C_F^{(k)} - C_G^{(k)}\big), \quad \forall 2 \le k \le n-1.$$

To get (3.5) for n, we note that

$$\begin{aligned}
& C_F^{(n)} - C_G^{(n)} \\
= \;& -n\int_a^b (x-\mu)^{n-1}[F(x)-G(x)]dx \\
= \;& n\int_a^b (x-\mu)^{n-1}[F_1^R(x)-G_1^R(x)]dx \\
= \;& n(x-\mu)^{n-1}[G_2^R(x)-F_2^R(x)]\Big|_a^b + \frac{n!}{(n-2)!}\int_a^b (x-\mu)^{n-2}[F_2^R(x)-G_2^R(x)]dx \\
= \;& \sum_{k=2}^{n} \frac{n!}{(n+1-k)!}\big(a-\mu\big)^{n+1-k}[F_k^R(a)-G_k^R(a)] + n![F_{n+1}^R(a)-G_{n+1}^R(a)]
\end{aligned}$$

Hence

$$\begin{aligned}
C_F^{(n)} - C_G^{(n)} &= n![F_{n+1}^R(a) - G_{n+1}^R(a)] \\
F_{n+1}(a) - G_{n+1}(a) &= \frac{1}{n!}(C_F^{(n)} - C_G^{(n)})
\end{aligned}$$

which is precisely (3.5) for n. Thus, the assertion of Part 1 of Theorem 3.2 holds. Part 2 and part 3 of Theorem 3.2 can be obtained from using the result from Part 1 of Theorem 3.2 holds. We turn to prove Part 4 of Theorem 3.2.

To prove Part 3, we will try to prove $F \succeq_n^R G$ implies $F_{n+1}^R(a) > G_{n+1}^R(a)$ under the assumption of Theorem 3.2. Since $F_n^R(x) \ge G_n^R(x)$ for each x and $F_n^R(x) > G_n^R(x)$ for at least one x_0 in $[a,b]$ implies $\int_a^b F_n^R(x)dx > \int_a^b G_n^R(x)dx$, that is $F_{n+1}^R(a) > G_{n+1}^R(a)$. Thus, the Part 3 of Theorem 3.2 holds. $\square$

Proof of Theorem 3.3.

By theorem 3.1, $G_{n+1}(b) \geq F_{n+1}(b)$ is equivalent to $(-1)^n C_G^{(n)} \geq (-1)^n C_F^{(n)}$ under the assumption of this theorem. Thus, we just need to prove that $G_{n+1}(b) \geq F_{n+1}(b)$ is equivalent to $Eu(F) \geq Eu(G)$ under the assumption of this theorem.

$$\begin{aligned}
\Delta Eu &\equiv Eu(F) - Eu(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= [G_2(b) - F_2(b)]u^{(1)}(b) - \int_a^b [G_2(x) - F_2(x)]u^{(2)}(x)dx \\
&= [G_2(b) - F_2(b)]u^{(1)}(b) - [G_3(b) - F_3(b)]u^{(2)}(b) + \int_a^b [G_3(x) - F_3(x)]u^{(3)}(x)dx \\
&= \sum_{k=2}^{n}(-1)^i[G_i(b) - F_i(b)]u^{(i-1)}(b) + (-1)^{n+1}\int_a^b [G_n(x) - F_n(x)]u^{(n)}(x)dx.
\end{aligned}$$

By our assumptions , Theorem 3.1 and Equation 3.2, we get $G_k(b) = F_k(b)$ for all $2 \leq k \leq n$. We have

$$\Delta Eu = (-1)^{n+1}\int_a^b [G_n(x) - F_n(x)]u^{(n)}(x)dx.$$

By the assumption, we have nonzero constant $u^{(n)}(x) = u^{(n)}$, thus

$$\Delta Eu = (-1)^{n+1}u^{(n)}[G_{n+1}(b) - F_{n+1}(b)].$$

Since $(-1)^{n+1}u^{(n)}$ is always positive, $[G_{n+1}(b) - F_{n+1}(b)] \geq 0$ implies $\Delta Eu \geq 0$ and $\Delta Eu \geq 0$ implies $[G_{n+1}(b) - F_{n+1}(b)] \geq 0$. Thus, the Theorem 3.3 holds. $\square$

Proof of Theorem 3.4.

By theorem 3.3, $F_{n+1}^R(a) \geq G_{n+1}^R(a)$ is equivalent to $C_F^{(n)} \geq C_G^{(n)}$ under the assumption of this theorem. Thus, we just need to prove that $F_{n+1}^R(a) \geq G_{n+1}^R(a)$ is equivalent to

$Eu(F) \geq Eu(G)$ under the assumption of this theorem.

$$\begin{aligned}
\Delta Eu &\equiv Eu(F) - Eu(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= \int_a^b [F_1^R(x) - G_1^R(x)]u^{(1)}(x)dx \\
&= [F_2^R(a) - G_2^R(a)]u^{(1)}(a) + \int_a^b [F_2^R(x) - G_2^R(x)]u^{(2)}(x)dx \\
&= [F_2^R(a) - G_2^R(a)]u^{(1)}(a) + [F_3^R(a) - G_3^R(a)]u^{(2)}(a) + \int_a^b [F_3^R(x) - G_3^R(x)]u^{(3)}(x)dx \\
&= \sum_{k=2}^{n} [F_i^R(a) - G_i^R(a)]u^{(i-1)}(a) + \int_a^b [F_n^R(x) - G_n^R(x)]u^{(n)}(x)dx.
\end{aligned}$$

By our assumptions, Theorem 3.3 and Equation 3.2, we get $F_k^R(a) = G_k^R(a)$ for all $2 \leq k \leq n$. We have

$$\Delta Eu = \int_a^b [F_n^R(x) - G_n^R(x)]u^{(n)}(x)dx.$$

By the assumption, we have nonzero constant $u^{(n)}(x) = u^{(n)}$, thus

$$\Delta Eu = u^{(n)}[F_{n+1}^R(a) - G_{n+1}^R(a)].$$

Since $u^{(n)}$ is always positive, $[F_{n+1}^R(a) - G_{n+1}^R(a)] \geq 0$ implies $\Delta Eu \geq 0$ and $\Delta Eu \geq 0$ implies $[F_{n+1}^R(a) - G_{n+1}^R(a)] \geq 0$. Thus, the Theorem 3.4 holds. $\square$

Proof of Theorems 4.3 and 4.4:

Using the Taylor formula, we can express the $u(F)$ as:

$$u(F) = u(\mu) + (x-\mu)u^{(1)}(\mu) + \frac{1}{2}(x-\mu)^2 u^{(2)}(\mu) + ... + \frac{1}{n!}(x-\mu)^n u^{(n)}(\mu) + R_X.$$

Assuming that the remainder R_X is negligible (or $u^{(n+1)} = 0$), then the expected utility could be expressed as:

$$Eu(F) = u(\mu) + \frac{1}{2}C_F^{(2)}u^{(2)}(\mu) + ... + \frac{1}{n!}C_F^{(n)}u^{(n)}(\mu) + R_n^F.$$

Similarly, for $u(G)$ we have:

$$Eu(G) = u(\mu) + \frac{1}{2}C_G^{(2)}u^{(2)}(\mu) + ... + \frac{1}{n!}C_G^{(n)}u^{(n)}(\mu) + R_n^G.$$

Under Theorem 4.3, $u^{(i)} = 0$ for any $i > n$ and thus the difference of expected utility between F and G could be expressed as:

$$Eu(F) - Eu(G) = \frac{1}{2}u^{(2)}(\mu)(C_F^{(2)} - C_G^{(2)}) + \cdots + \frac{1}{n!}u^{(n)}(\mu)(C_F^{(n)} - C_G^{(n)}).$$

Thus, $X\ M_{RA}^n\ Y$ or $X\ M_{RS}^n\ Y$ implies $Eu(F) - Eu(G) \geq 0$ under its corresponding utility assumption.

Under Theorem 4.4, both R_n^F and R_n^G are ignorable. Hence,

$$Eu(F) - Eu(G) \approx \frac{1}{2}u^{(2)}(\mu)(C_F^{(2)} - C_G^{(2)}) + \cdots + \frac{1}{n!}u^{(n)}(\mu)(C_F^{(n)} - C_G^{(n)}),$$

and thus, $X\ M_{RA}^n\ Y$ or $X\ M_{RS}^n\ Y$ implies $Eu(F) - Eu(G) \geq 0$ under its corresponding utility assumption.

Proof of Theorem 4.5:

Consider first the M_{RA}^2 rule. Assume that $u^{(3)} = 0$. Then we have $u^{(2)}(\mu_X) = u^{(2)}(\mu_Y) = c < 0$ and further:

$$\begin{aligned} Eu(F) &= u(\mu_X) + \frac{1}{2}C_F^{(2)}u^{(2)}(\mu_X); \\ Eu(G) &= u(\mu_Y) + \frac{1}{2}C_G^{(2)}u^{(2)}(\mu_Y). \end{aligned}$$

Thus the difference of expected utility between F and G could be expressed as:

$$Eu(F) - Eu(G) = u(\mu_X) - u(\mu_Y) + \frac{1}{2}[C_F^{(2)} - C_G^{(2)}]c.$$

Since XM_{RA}^2Y, we have $\mu_X \geq \mu_Y$ and $C_F^{(2)} \leq C_G^{(2)}$, thus we have $Eu(F) \geq Eu(G)$.

Now we turn to consider M_{RA}^3 rule. Assume now that $u^{(4)} = 0$. Then we have $u^{(3)}(\mu_X) = u^{(3)}(\mu_Y) = c > 0$ and further:

$$\begin{aligned} Eu(F) &= u(\mu_X) + \frac{1}{2}C_F^{(2)}u^{(2)}(\mu_X) + \frac{1}{3!}C_F^{(3)}u^{(3)}(\mu_X); \\ Eu(G) &= u(\mu_Y) + \frac{1}{2}C_G^{(2)}u^{(2)}(\mu_Y) + \frac{1}{3!}C_F^{(3)}u^{(3)}(\mu_Y). \end{aligned}$$

Thus the difference of expected utility between F and G could be expressed as:

$$Eu(F) - Eu(G) = u(\mu_X) - u(\mu_Y) + \frac{1}{2}[C_F^{(2)}u^{(2)}(\mu_X) - C_G^{(2)}u^{(2)}(\mu_Y)] + \frac{1}{3!}c(C_F^{(3)} - C_G^{(3)}).$$

Since $u^{(3)} > 0$ and $\mu_X \geq \mu_Y$, we have $u^{(2)}(\mu_X) \geq u^{(2)}(\mu_Y)$. It follows that

$$\begin{aligned} & C_F^{(2)} u^{(2)}(\mu_X) - C_G^{(2)} u^{(2)}(\mu_Y) \\ = \; & C_F^{(2)}(u^{(2)}(\mu_X) - u^{(2)}(\mu_Y)) + u^{(2)}(\mu_Y)(C_F^{(2)} - C_G^{(2)}) \geq 0. \end{aligned}$$

Since $XM_{RA}^3 Y$, we have $\mu_X \geq \mu_Y$, $0 \leq C_F^{(2)} \leq C_G^{(2)}$ and $C_F^{(3)} \geq C_G^{(3)}$, thus we have $Eu(F) \geq Eu(G)$. The proofs for Part 2 in Theorem 4.5 are similar and thus omitted here. □

Proof of Theorem 4.6:

Similar to the derivations in the proof for Theorem 4.5, we can have:

$$\begin{aligned} Eu(F) - Eu(G) \; = \; & u(\mu_X) - u(\mu_Y) + \frac{1}{2}[C_F^{(2)} u^{(2)}(\mu_X) - C_G^{(2)} u^{(2)}(\mu_Y)] + \cdots \\ & + \frac{1}{n!}[C_F^{(n)} u^{(n)}(\mu_X) - C_G^{(n)} u^{(n)}(\mu_Y)]. \end{aligned}$$

We need to consider the sign of the terms $A_i = C_F^{(i)} u^{(i)}(\mu_X) - C_G^{(i)} u^{(i)}(\mu_Y), i = 2, \cdots, n$. First consider the even numbers i. In this situation, we have

$$u^{(i)} \leq 0, u^{(i+1)} \geq 0, C_F^{(i)} \leq C_G^{(i)} \text{ and } C_F^{(i)} \geq 0.$$

Then we get:

$$A_i \; = \; C_F^{(i)}(u^{(i)}(\mu_X) - u^{(i)}(\mu_Y)) + u^{(i)}(\mu_Y)(C_F^{(i)} - C_G^{(i)}) \geq 0.$$

Now assume that i is odd number. We have:

$$u^{(i)} \geq 0, u^{(i+1)} \leq 0, C_F^{(i)} \geq C_G^{(i)}.$$

Then we get:

$$A_i \; = \; u^{(i)}(\mu_X)(C_F^{(i)} - C_G^{(i)}) + C_G^{(i)}(u^{(i)}(\mu_X) - u^{(i)}(\mu_Y)).$$

A sufficient condition for A_i being positive is that $C_G^{(i)} \leq 0$. As a result, Part 1 of Theorem 4.6 holds. The proofs for Part 2 in Theorem 4.6 are similar and thus omitted here. □

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