Strang-type Preconditioners for Solving System of Delay Differential Equations by Boundary Value Methods

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Abstract

In this paper, we survey some of the latest developments in using boundary value methods (BVMs) for solving systems of delay differential equations (DDEs). These methods require the solutions of nonsymmetric, large and sparse linear systems. The GMRES method with the Strang-type preconditioner is proposed for solving these systems. One of the main results is that if an A_{ν_1,ν_2}-stable BVM is used for a system of DDEs, then the preconditioner is invertible and the preconditioned matrix can be decomposed as $I + L$ where I is the identity matrix and L is a low rank matrix. It follows that when the GMRES method is applied to solving the preconditioned systems, the method will converge fast.

1 Introduction

Let us begin with the initial value problem (IVP) of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = J_m \mathbf{y}(t) + \mathbf{g}(t), \quad t \in (t_0, T], \\ \\ \mathbf{y}(t_0) = \mathbf{z}, \end{cases} \tag{1}$$

where $\mathbf{y}(t)$, $\mathbf{g}(t) : \mathbb{R} \to \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^m$, and J_m is the stiffness matrix in $\mathbb{R}^{m\times m}$. The initial value methods (IVMs), such as the Runge-Kutta methods, are well-known methods for solving (1), see [18]. Recently, another class of methods called the boundary value methods (BVMs) has been proposed, see [5]. Using BVMs to discretize (1), we get a linear system $M\mathbf{u} = \mathbf{b}$ where $\mathbf{u}$, $\mathbf{b}$ are vectors and M is a matrix depending on the multistep rule we used. The advantage of using BVMs is that the methods are more stable and the resulting linear system is hence more well-conditioned. However, the system is in general large and sparse (with band-structure), and solving it is a major problem in the application of the BVMs. We use the GMRES method, which is one of Krylov subspace methods, for solving the discrete system. In order to speed up the convergence of the GMRES iterations, the Strang-type preconditioner S is proposed to precondition the discrete

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system. The advantage of the Strang-type preconditioner is that if an A_{ν_1,ν_2}-stable BVM is used in (1), then S is invertible and the preconditioned matrix can be decomposed as

$$S^{-1}M = I + L,$$

where the rank of L is at most $2m(\nu_1+\nu_2)$ which is independent of the integration step size. It follows that the GMRES method applied to the preconditioned system will converge in at most $2m(\nu_1+\nu_2)+1$ iterations in exact arithmetic.

BVMs for the solution of ordinary differential equations of the form (1) have been studied in [6]. In this paper, we discuss the use of BVMs to solve three different types of delay differential equations. The GMRES method with the Strang-type preconditioners is proposed to solve the resulting linear systems. The outline of the paper is as follows. In §2, we give some background knowledge about the linear multistep formulae. Then, we investigate the properties of the Strang-type block-circulant preconditioners for the neutral delay differential equations, the differential equations with multi-delays, and the singular perturbation delay differential equations in §3–§5, respectively. The convergence analysis of the method is given with some illustrative numerical examples.

2 Linear Multistep Formulae

Consider a single 1-dimensional IVP

$$\begin{cases} y' = f(t,y), & t \in (t_0, T], \\ y(t_0) = y_0. \end{cases} \tag{2}$$

Over a uniform mesh

$$t_j = t_0 + jh, \qquad j = 0, \cdots, r,$$

with step size $h = (T-t_0)/r$, the μ-step *linear multistep formula* (LMF) is defined as follows:

$$\sum_{j=0}^{\mu} \alpha_j y_{n+j} - h \sum_{j=0}^{\mu} \beta_j f_{n+j} = 0, \qquad n = 0, \cdots, r-\mu. \tag{3}$$

Here, y_n is the discrete approximation to $y(t_n)$, $f_n = f(t_n, y_n)$.

To get the solution by (3), we need μ initial conditions y_0, y_1, $\cdots$, $y_{\mu-1}$. Since only y_0 is provided from the original problem, we have to find additional conditions for the remaining values y_1, y_2, $\cdots$,, $y_{\mu-1}$. The method in (3) with the $(\mu-1)$ additional conditions is called *Initial Value Methods* (IVMs). An IVM is called *implicit* if $\beta_\mu \neq 0$ and *explicit* if $\beta_\mu = 0$. If an IVM is applied to the IVP (2) on the interval $[t_0, t_{r+\mu-1}]$ with uniform stepsize h, we have the following discrete problem

$$A_r \mathbf{y} = h B_r \mathbf{f} + \mathbf{g} \tag{4}$$

where $\mathbf{y} = (y_\mu, y_{\mu+1}, \cdots, y_{r+\mu-1})^T$, $\mathbf{f} = (f_\mu, f_{\mu+1}, \cdots, f_{r+\mu-1})^T$,

$$\mathbf{g} = (\sum_{i=0}^{\mu-1}(\alpha_i y_i - h\beta_i f_i), \cdots, \alpha_0 y_{\mu-1} - h\beta_0 f_{\mu-1}, 0, \cdots, 0)^T,$$

$$A_r = \begin{pmatrix} \alpha_\mu & & & & \\ \vdots & \ddots & & & \\ \alpha_0 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \alpha_0 & \cdots & \alpha_\mu \end{pmatrix}_{r\times r}, \qquad B_r = \begin{pmatrix} \beta_\mu & & & & \\ \vdots & \ddots & & & \\ \beta_0 & & \ddots & & \\ & \ddots & & \ddots & \\ & & \beta_0 & \cdots & \beta_\mu \end{pmatrix}_{r\times r}.$$

Note that the matrices A_r and B_r are lower triangular Toeplitz matrices. We recall that a matrix is said to be Toeplitz if its entries are constant along its diagonals. Moreover, the linear system (4) can be solved easily by forward recursion. A classical example of IVM is the second order backward differentiation formulae:

$$3y_{n+2} - 4y_{n+1} + y_n = 2hf_{n+1},$$

which is a two-step method with $\alpha_0 = 1$, $\alpha_1 = -4$, $\alpha_2 = 3$ and $\beta_0 = 2$.

Instead of using IVM which sets μ initial conditions for (3), we can also use the so-called *Boundary Value Methods* (BVMs). Given ν_1, $\nu_2 \geq 0$ such that $\nu_1 + \nu_2 = \mu$, then the corresponding BVM requires ν_1 initial addition conditions y_0, y_1, $\cdots$, $y_{\nu_1 - 1}$ and ν_2 final addition conditions y_r, y_{r+1}, $\cdots$, $y_{r+\nu_2-1}$, which are called (ν_1, ν_2)-boundary conditions. Note that the class of BVMs contains the class of IVMs (i.e. $\nu_1 = \mu$, $\nu_2 = 0$).

The discrete problem generated by a μ-step BVM with (ν_1, ν_2)-boundary conditions can be written in the following matrix form

$$A\mathbf{y} = hB\mathbf{f} + \mathbf{g}$$

where $\mathbf{y} = (y_{\nu_1}, y_{\nu_1+1}, \cdots, y_{r-1})^T$, $\mathbf{f} = (f_{\nu_1}, f_{\nu_1+1}, \cdots, f_{r-1})^T$,

$$\begin{aligned} \mathbf{g} \;\; = \;\; & \left(\sum_{i=0}^{\nu_1 - 1} (\alpha_i y_i - h\beta_i f_i), \cdots, \alpha_0 y_{\nu_1-1} - h\beta_0 f_{\nu_1 - 1}, 0, \cdots, 0, \right. \\ & \qquad\qquad \left. \alpha_\mu y_r - h\beta_\mu f_r, \cdots, \sum_{i=1}^{\nu_2} (\alpha_{\nu_1+i} y_{r-1+i} - h\beta_{\nu_1+i} f_{r-1+i}) \right)^T, \end{aligned}$$

A and $B \in \mathbb{R}^{(r-\nu_1)\times(r-\nu_1)}$ are defined as follows,

$$A = \begin{pmatrix} \alpha_{\nu_1} & \cdots & \alpha_\mu & & \\ \vdots & \ddots & \ddots & \ddots & \\ \alpha_0 & \ddots & \ddots & \ddots & \alpha_\mu \\ & \ddots & & \ddots & \vdots \\ & & \alpha_0 & \cdots & \alpha_{\nu_1} \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_{\nu_1} & \cdots & \beta_\mu & & \\ \vdots & \ddots & \ddots & \ddots & \\ \beta_0 & \ddots & \ddots & \ddots & \beta_\mu \\ & \ddots & & \ddots & \vdots \\ & & \beta_0 & \cdots & \beta_{\nu_1} \end{pmatrix}. \tag{5}$$

Note that the coefficient matrices are Toeplitz with lower bandwidth ν_1 and upper bandwidth ν_2. An example of BVMs is the third order generalized backward differentiation formulae (GBDF),

$$2y_{n+1} + 3y_n - 6y_{n-1} + y_{n-2} = 6hf_n,$$

which is a three-step method with (2,1)-boundary conditions, and that $\alpha_0 = 1$, $\alpha_1 = -6$, $\alpha_2 = 3$, $\alpha_4 = 2$ and $\beta_0 = 6$.

In order to study the stability properties of the BVM, we introduce the characteristic polynomials $\rho(z)$ and $\sigma(z)$ of the BVM which are defined by

$$\rho(z) \equiv \sum_{j=0}^{\mu} \alpha_j z^j \quad \text{and} \quad \sigma(z) \equiv \sum_{j=0}^{\mu} \beta_j z^j, \tag{6}$$

where $\{\alpha_i\}_{i=0}^{\mu}$ and $\{\beta_i\}_{i=0}^{\mu}$ are given by (3). The A_{ν_1,ν_2}-stability polynomial is defined by

$$\pi(z,q) \equiv \rho(z) - q\sigma(z) \tag{7}$$

where $z,\, q \in \mathbb{C}$. Let $\mathbb{C}^- \equiv \{q \in \mathbb{C} : \mathrm{Re}(q) < 0\}$.

Definition 1 **[5]** *The region*

$$\mathcal{D}_{\nu_1,\nu_2} = \{q \in \mathbb{C} : \pi(z,q) \text{ has } \nu_1 \text{ zeros inside } |z| = 1 \text{ and } \nu_2 \text{ zeros outside } |z| = 1\}$$

is called the region of A_{ν_1,ν_2}-stability of a given BVM with (ν_1, ν_2)-boundary conditions. Moreover, the BVM is said to be A_{ν_1,ν_2}-stable if

$$\mathbb{C}^- \subseteq \mathcal{D}_{\nu_1,\nu_2}.$$

Although IVMs are more efficient than BVMs (which cannot be solved by forward recursion), the advantage in using BVMs over IVMs comes from their stability properties. For example, the usual backward differentiation formulae are not A-stable for $\mu > 2$ but the GBDF are $A_{\nu,\mu-\nu}$-stable for any $\mu \geq 1$, see for instances [1, 3] and [5, p. 79 and Figures 5.1–5.3].

3 Neutral Delay Differential Equations

In this section, we consider the solution of neutral delay differential equations:

$$\begin{cases} \mathbf{y}'(t) = L_n\mathbf{y}'(t-\tau) + M_n\mathbf{y}(t) + N_n\mathbf{y}(t-\tau), & t \geq t_0, \\ \\ \mathbf{y}(t) = \boldsymbol{\phi}(t), & t \leq t_0, \end{cases} \tag{8}$$

where $\mathbf{y}(t)$, $\boldsymbol{\phi}(t) : \mathbb{R} \to \mathbb{R}^n$; L_n, M_n, $N_n \in \mathbb{R}^{n\times n}$, and $\tau > 0$ is a constant. By applying a BVM, the discrete solution of (8) is given by the solution of a linear system

$$H\mathbf{y} = \mathbf{b}$$

where H depends on the LMF used.

3.1 BVMs and Their Stability Properties

In order to find a reasonable numerical solution, we require that the solution of (8) is asymptotically stable. Let $\lambda(\cdot)$ denote the spectrum of a matrix, I_n the n-by-n identity matrix, and $\|\cdot\|_2$ the 2-norm. We have the following lemma, see [9, 13].

Lemma 1 *Let L_n, M_n and N_n be any matrices and $\|L_n\|_2 < 1$. Then the solution of (8) is asymptotically stable if $Re(\lambda_i) < 0$ for any i, where*

$$\lambda_i \in \lambda\left((I_n - \eta L_n)^{-1}(M_n + \eta N_n)\right)$$

with $|\eta| \leq 1$.

Let $h = \tau/k_1$ be the step size where k_1 is a positive integer. For (8), by using a BVM with (ν_1, ν_2)-boundary conditions over a uniform mesh

$$t_j = t_0 + jh, \qquad j = 0, \cdots, r_1,$$

on the interval $[t_0, t_0 + r_1 h]$, we have

$$\sum_{i=0}^{\mu} \alpha_i \mathbf{y}_{p+i-\nu_1} = \sum_{i=0}^{\mu} \alpha_i L_n \mathbf{y}_{p+i-\nu_1-k_1} + h \sum_{i=0}^{\mu} \beta_i (M_n \mathbf{y}_{p+i-\nu_1} + N_n \mathbf{y}_{p+i-\nu_1-k_1}), \tag{9}$$

for $p = \nu_1, \ldots, r_1 - 1$, where $\mu = \nu_1 + \nu_2$, and $\{\alpha_i\}_{i=0}^{\mu}$, $\{\beta_i\}_{i=0}^{\mu}$ are the coefficients of the given BVM, see [2]. By providing the values

$$\mathbf{y}_{-k_1}, \ldots, \mathbf{y}_0, \quad \mathbf{y}_1, \ldots, \mathbf{y}_{\nu_1-1}, \quad \mathbf{y}_{r_1}, \ldots, \mathbf{y}_{r_1+\nu_2-1}, \tag{10}$$

(9) can be written in a matrix form as

$$H\mathbf{y} = \mathbf{b}$$

where

$$H \equiv A \otimes I_n - A^{(1)} \otimes L_n - hB \otimes M_n - hB^{(1)} \otimes N_n, \tag{11}$$

$$\mathbf{y}^T = (\mathbf{y}_{\nu_1}^T, \mathbf{y}_{\nu_1+1}^T, \cdots, \mathbf{y}_{r_1-1}^T) \in \mathbb{R}^{n(r_1-\nu_1)},$$

and $\mathbf{b} \in \mathbb{R}^{n(r_1-\nu_1)}$ is a vector that depends on the boundary values and the coefficients of the method. In (11), the matrices A, $B \in \mathbb{R}^{(r_1-\nu_1)\times(r_1-\nu_1)}$ are defined as in (5), and $A^{(1)}$, $B^{(1)} \in \mathbb{R}^{(r_1-\nu_1)\times(r_1-\nu_1)}$ are given as follows:

$$A^{(1)} = \begin{pmatrix} \mathbf{0} & & & & & \\ \alpha_\mu & \ddots & & & & \\ \vdots & \ddots & \ddots & & & \\ \alpha_0 & \cdots & \alpha_\mu & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ & & \alpha_0 & \cdots & \alpha_\mu & \mathbf{0} \end{pmatrix}, \qquad B^{(1)} = \begin{pmatrix} \mathbf{0} & & & & & \\ \beta_\mu & \ddots & & & & \\ \vdots & \ddots & \ddots & & & \\ \beta_0 & \cdots & \beta_\mu & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ & & \beta_0 & \cdots & \beta_\mu & \mathbf{0} \end{pmatrix}, \tag{12}$$

see [2]. We remark that the first column of $A^{(1)}$ is given by:

$$(\underbrace{0, \ldots, 0}_{k_1+\nu_1-\mu}, \alpha_\mu, \ldots, \alpha_0, \underbrace{0, \ldots, 0}_{r_1-k_1-2\nu_1-1})^T,$$

and the first column of $B^{(1)}$ is given by

$$(\underbrace{0, \ldots, 0}_{k_1+\nu_1-\mu}, \beta_\mu, \ldots, \beta_0, \underbrace{0, \ldots, 0}_{r_1-k_1-2\nu_1-1})^T.$$

3.2 Strang-type Preconditioner

We first define the Strang's circulant preconditioner for Toeplitz matrix. For any given Toeplitz matrix $T_l = [t_{i-j}]_{i,j=1}^{l} = [t_q]$, Strang's preconditioner $s(T_l)$ is a circulant matrix with diagonals given by

$$[s(T_l)]_q = \begin{cases} t_q, & 0 \le q \le \lfloor l/2 \rfloor, \\ \\ t_{q-l}, & \lfloor l/2 \rfloor < q < l, \\ \\ [s(T_l)]_{l+q}, & 0 < -q < l, \end{cases}$$

see [7, 10]. The Strang-type block-circulant (BC) preconditioner for (11) is defined as follows:

$$\begin{aligned} \bar{S} &\equiv s(A) \otimes I_n - s(A^{(1)}) \otimes L_n - hs(B) \otimes M_n - hs(B^{(1)}) \otimes N_n && (13) \\ &= (F^* \otimes I_n)(\Lambda_A \otimes I_n - \Lambda_{A^{(1)}} \otimes L_n - h\Lambda_B \otimes M_n - h\Lambda_{B^{(1)}} \otimes N_n)(F \otimes I_n), && (14) \end{aligned}$$

where $s(E)$ is Strang's preconditioner of Toeplitz matrix E, Λ_E is the diagonal matrix given by

$$\Lambda_E = F s(E) F^*,$$

for $E = A$, B, $A^{(1)}$, $B^{(1)}$ respectively, and F is the Fourier matrix.

Now we discuss the invertibility of the Strang-type preconditioner. Let $w_j = e^{\frac{2\pi i j}{r_1 - \nu_1}}$ where $i \equiv \sqrt{-1}$, we have

$$[\Lambda_A]_{jj} = \rho(w_j)/w_j^{\nu_1}, \qquad [\Lambda_B]_{jj} = \sigma(w_j)/w_j^{\nu_1},$$

$$[\Lambda_{A^{(1)}}]_{jj} = \alpha_\mu w_j^{-k_1-\nu_1+\mu} + \cdots + \alpha_0 w_j^{-k_1-\nu_1} = \rho(w_j)/w_j^{k_1+\nu_1},$$

and

$$[\Lambda_{B^{(1)}}]_{jj} = \beta_\mu w_j^{-k_1-\nu_1+\mu} + \cdots + \beta_0 w_j^{-k_1-\nu_1} = \sigma(w_j)/w_j^{k_1+\nu_1},$$

where $\rho(z)$ and $\sigma(z)$ are defined as in (6). Thus the jth-block of

$$\Lambda_A \otimes I_n - \Lambda_{A^{(1)}} \otimes L_n - h\Lambda_B \otimes M_n - h\Lambda_{B^{(1)}} \otimes N_n$$

in (14) is given by

$$\begin{aligned} \bar{S}_j &= [\Lambda_A]_{jj} I_n - [\Lambda_{A^{(1)}}]_{jj} L_n - h[\Lambda_B]_{jj} M_n - h[\Lambda_{B^{(1)}}]_{jj} N_n \\ &= \frac{1}{w_j^{k_1+\nu_1}} \Big[w_j^{k_1}(\rho(w_j) I_n - h\sigma(w_j) M_n) - \rho(w_j) L_n - h\sigma(w_j) N_n \Big], \end{aligned}$$

for $j = 1, 2, \dots, r_1 - \nu_1$. In order to prove that $\bar{S}$ is invertible, we need to establish that $\bar{S}_j$ is invertible for $j = 1, 2, \dots, r_1 - \nu_1$. Let

$$\begin{aligned} \Delta &\equiv e^{ik_1\theta}(\rho(e^{i\theta}) I_n - h\sigma(e^{i\theta}) M_n) - \rho(e^{i\theta}) L_n - h\sigma(e^{i\theta}) N_n \\ &= e^{ik_1\theta}(I_n - e^{-ik_1\theta} L_n) D \end{aligned}$$

where

$$D \equiv \rho(e^{i\theta}) I_n - h(I_n - e^{-ik_1\theta} L_n)^{-1}(M_n + e^{-ik_1\theta} N_n)\sigma(e^{i\theta}). \qquad (15)$$

So, to prove that $\bar{S}_j$ is invertible, we only need to show that Δ is invertible for any $\theta \in \mathbb{R}$. Assume that $\|L_n\|_2 < 1$, we have $I_n - e^{-ik_1\theta} L_n$ is nonsingular for any $\theta \in \mathbb{R}$. Therefore, we only need to show for any $\theta \in \mathbb{R}$, D is invertible. We have the following theorem.

Theorem 1 **[2]** *If the BVM with (ν_1, ν_2)-boundary conditions is A_{ν_1,ν_2}-stable and the conditions in Lemma 1 hold, then for any $\theta \in \mathbb{R}$, the matrix D defined by (15) is invertible. It follows that the Strang-type preconditioner $\bar{S}$ defined as in (13) is also invertible.*

3.3 Spectral Analysis and Operation Cost

In this section, we discuss the convergence rate of the preconditioned GMRES method with the Strang-type BC preconditioner. First of all, we recall the following well-known result.

Lemma 2 **[10]** *Let W be invertible and can be decomposed as $W = I + L$. If the GMRES method is applied to solving the linear system $W\mathbf{x} = \mathbf{b}$, then the method will converge in at most* $\mathrm{rank}(L) + 1$ *iterations in exact arithmetic.*

Using Lemma 2, we have the following result for the spectra of preconditioned matrices and for the convergence rate of our method.

Theorem 2 **[2]** *Let H be given by (11) and $\bar{S}$ be given by (13), then we have*

$$\bar{S}^{-1}H = I_{n(r_1-\nu_1)} + L$$

where $I_{n(r_1-\nu_1)} \in \mathbb{R}^{n(r_1-\nu_1)\times n(r_1-\nu_1)}$ is the identity matrix and L is a low rank matrix with

$$\mathrm{rank}(L) \leq 2(\mu + k_1 + \nu_1 + 1)n.$$

Therefore, when the GMRES method is applied to solving the preconditioned system

$$\bar{S}^{-1}H\mathbf{y} = \bar{S}^{-1}\mathbf{b},$$

the method will converge in at most $2(\mu + k_1 + \nu_1 + 1)n + 1$ iterations in exact arithmetic.

We observe from Theorem 2 that if the step size $h = \tau/k_1$ is fixed, the number of iterations for convergence of the GMRES method, when applied to solving $\bar{S}^{-1}H\mathbf{y} = \bar{S}^{-1}\mathbf{b}$, will be independent of r_1, that is to say, it is independent of the length of the interval that we considered.

Regarding the cost per iteration, the main work in each GMRES iteration is the matrix-vector multiplication $\bar{S}^{-1}H\mathbf{y}$. Since $A, A^{(1)}, B, B^{(1)}$ are band matrices and L_n, M_n, N_n are assumed to be sparse, the matrix–vector multiplication $H\mathbf{y}$ can be done very fast. To compute $\bar{S}^{-1}\mathbf{z}$ for any vector $\mathbf{z}$, by (14), it follows that

$$\bar{S}^{-1}\mathbf{z} = (F^* \otimes I_n)(\Lambda_A \otimes I_n - \Lambda_{A^{(1)}} \otimes L_n - h\Lambda_B \otimes M_n - h\Lambda_{B^{(1)}} \otimes N_n)^{-1}(F \otimes I_n)\mathbf{z}.$$

This product can be obtained by using Fast Fourier Transforms and solving $r_1 - \nu_1$ linear systems of order n, see [4]. Since L_n, M_n, N_n are sparse, the coefficient matrices of the n-by-n linear systems will also be sparse. Thus $\bar{S}^{-1}\mathbf{z}$ can be obtained by solving $r_1 - \nu_1$ sparse n-by-n linear systems.

3.4 Numerical Result

We illustrate the efficiency of our preconditioner by solving Example 1 below. We used the MATLAB-provided M-file "gmres" (see MATLAB on-line documentation) to solve the preconditioned systems. In all of our tests, the zero vector is the initial guess and the stopping criterion is

$$\frac{||\mathbf{r}_q||_2}{||\mathbf{r}_0||_2} < 10^{-6}$$

where $\mathbf{r}_q$ is the residual after the q-th iteration.

Example 1. Consider

$$\begin{cases} \mathbf{y}'(t) = L_n\mathbf{y}'(t-1) + M_n\mathbf{y}t + N_n\mathbf{y}(t-1), & t \geq 0, \\ \mathbf{y}(t) = (1, 1, \cdots, 1)^T, & t \leq 0, \end{cases}$$

where

$$L_n = \frac{1}{n}\begin{pmatrix} 2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{pmatrix}, \quad M_n = \begin{pmatrix} -8 & 2 & 1 & & \\ 2 & \ddots & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & 1 \\ & \ddots & \ddots & \ddots & 2 \\ & & 1 & 2 & -8 \end{pmatrix},$$

and

$$N_n = \frac{1}{n}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}.$$

Example 1 is solved by using the fifth order generalized Adams method for $t \in [0, 4]$. Table 1 lists the number of iterations required for convergence of the GMRES method for different n and k_1. In the table, $\mathcal{I}$ means no preconditioner is used and $\bar{S}$ denotes the Strang-type BC preconditioner defined in (13). We see that the number of iterations required for convergence, when a circulant preconditioner is used, is always less than that when no preconditioner is used. We should emphasize that our numerical example shows a much faster convergence rate than that predicted by the estimate provided by Theorem 2.

4 Differential Equations with Multi-delays

Consider the solution of differential equations with multi-delays:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = J_n\mathbf{y}(t) + D_n^{(1)}\mathbf{y}(t-\tau_1) + \cdots + D_n^{(s)}\mathbf{y}(t-\tau_s) + \mathbf{f}(t), & t \geq t_0, \\ \mathbf{y}(t) = \boldsymbol{\phi}(t), & t \leq t_0, \end{cases} \tag{16}$$

where $\mathbf{y}(t)$, $\mathbf{f}(t)$, $\boldsymbol{\phi}(t) : \mathbb{R} \to \mathbb{R}^n$; J_n, $D_n^{(1)}, \cdots, D_n^{(s)} \in \mathbb{R}^{n\times n}$ and $\tau_1, \cdots, \tau_s > 0$ are some rational numbers.

n	k_1	$\mathcal{I}$	$\bar{S}$
24	10	43	7
	20	83	7
	40	161	7
	80	*	7

n	k_1	$\mathcal{I}$	$\bar{S}$
48	10	44	6
	20	83	6
	40	163	6
	80	*	6

Table 1: Number of iterations for convergence ('$*$' means out of memory).

4.1 BVMs and Strang-type Preconditioner

As in §3.1, we want to find an asymptotically stable solution for (16). We have the following lemma, see [15, 19].

Lemma 3 *For any $s \geq 1$, if $\eta(J_n) \equiv \frac{1}{2}\lambda_{\max}(J_n + J_n^T) < 0$ and*

$$\eta(J_n) + \sum_{j=1}^{s} ||D_n^{(j)}||_2 < 0, \tag{17}$$

then solution of (16) is asymptotically stable.

In the following, for simplicity, we only consider the case of $s = 2$ in (16). The generalization to arbitrary s is straightforward. Let

$$h = \tau_1/m_1 = \tau_2/m_2$$

be the step size where m_1 and m_2 are positive integers with $m_2 > m_1$ ($\tau_2 > \tau_1$). For (16), by using a BVM with (ν_1, ν_2)-boundary conditions over a uniform mesh

$$t_j = t_0 + jh, \qquad j = 0, \cdots, r_2,$$

on the interval $[t_0, t_0 + r_2 h]$, we have

$$\sum_{i=0}^{\mu} \alpha_i \mathbf{y}_{p+i-\nu_1} = h \sum_{i=0}^{\mu} \beta_i (J_n \mathbf{y}_{p+i-\nu_1} + D_n^{(1)} \mathbf{y}_{p+i-\nu_1-m_1} + D_n^{(2)} \mathbf{y}_{p+i-\nu_1-m_2} + \mathbf{f}_{p+i-\nu_1}), \tag{18}$$

for $p = \nu_1, \ldots, r_2 - 1$, where $\mu = \nu_1 + \nu_2$, and $\{\alpha_i\}_{i=0}^{\mu}$, $\{\beta_i\}_{i=0}^{\mu}$ are the coefficients of the given BVM.

By providing the values

$$y_{-m_2}, \cdots, y_{-m_1}, \cdots, y_0, \quad y_1, \cdots, y_{\nu_1-1}, \quad y_{r_2}, \cdots, y_{r_2+\nu_2-1}, \tag{19}$$

(18) can be written in a matrix form as

$$R\mathbf{y} = \mathbf{b} \tag{20}$$

where

$$R \equiv A \otimes I_n - hB \otimes J_n - hC^{(1)} \otimes D_n^{(1)} - hC^{(2)} \otimes D_n^{(2)}, \tag{21}$$

$$\mathbf{y}^T = (\mathbf{y}_{\nu_1}^T, \mathbf{y}_{\nu_1+1}^T, \cdots, \mathbf{y}_{r_2-1}^T) \in \mathbb{R}^{n(r_2-\nu_1)},$$

and $\mathbf{b} \in \mathbb{R}^{n(r_2-\nu_1)}$ depends on $\mathbf{f}$, the boundary values, and the coefficients of the method. The matrices A, $B \in \mathbb{R}^{(r_2-\nu_1)\times(r_2-\nu_1)}$ are defined as in (5) and $C^{(1)}$, $C^{(2)} \in \mathbb{R}^{(r_2-\nu_1)\times(r_2-\nu_1)}$ in (21) are defined as the matrix $B^{(1)}$ in (12). We remark that the first column of $C^{(1)}$ is given by

$$(\underbrace{0, \cdots, 0}_{m_1+\nu_1-\mu}, \beta_\mu, \cdots, \beta_0, \underbrace{0, \cdots, 0}_{r_2-m_1-2\nu_1-1})^T$$

and the first column of $C^{(2)}$ is given by

$$(\underbrace{0, \cdots, 0}_{m_2+\nu_1-\mu}, \beta_\mu, \cdots, \beta_0, \underbrace{0, \cdots, 0}_{r_2-m_2-2\nu_1-1})^T.$$

The Strang-type BC preconditioner for (21) is defined as follows:

$$\tilde{S} \equiv s(A) \otimes I_n - hs(B) \otimes J_n - hs(C^{(1)}) \otimes D_n^{(1)} - hs(C^{(2)}) \otimes D_n^{(2)} \tag{22}$$

where $s(E)$ is Strang's circulant preconditioner of matrix E, for $E = A$, B, $C^{(1)}$ and $C^{(2)}$. We have the following theorem for the invertibility of our preconditioner $\tilde{S}$ and for the convergence rate of our method.

Theorem 3 **[11]** *If the BVM for (16) is A_{ν_1,ν_2}-stable and (17) holds, the Strang-type BC preconditioner $\tilde{S}$ defined as in (22) is invertible. Moreover, when the GMRES method is applied to solving the preconditioned system*

$$\tilde{S}^{-1}R\mathbf{y} = \tilde{S}^{-1}\mathbf{b},$$

the methods will converge in at most $(2\mu+m_1+m_2+2\nu_1+2)n = \mathcal{O}(n)$ iterations in exact arithmetic.

We know from Theorem 3 that if the step size $h = \tau_1/m_1 = \tau_2/m_2$ is fixed, the number of iterations for convergence of the GMRES method, when applied to the preconditioned system $\tilde{S}^{-1}R\mathbf{y} = \tilde{S}^{-1}\mathbf{b}$, will be independent of r_2 and therefore is independent of the length of the interval that we considered.

For the operation cost of our algorithm, we refer to [11, 14].

4.2 Numerical Test

We illustrate the efficiency of our preconditioner by solving the following problem. In the example, the BVM we used is the third order GBDF for $t \in [0, 4]$.

Example 2. Consider

$$\begin{cases} \mathbf{y}'(t) = J_n\mathbf{y}(t) + D_n^{(1)}\mathbf{y}(t-0.5) + D_n^{(2)}\mathbf{y}(t-1), & t \geq 0, \\ \mathbf{y}(t) = (\sin t, 1, \cdots, 1)^T, & t \leq 0, \end{cases}$$

where

$$J_n = \begin{pmatrix} -10 & 2 & & & \\ 2 & \ddots & \ddots & & \\ 1 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & 2 \\ & & 1 & 2 & -10 \end{pmatrix}, \qquad D_n^{(1)} = \frac{1}{n}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix},$$

and

$$D_n^{(2)} = \frac{1}{n} \begin{pmatrix} 2 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{pmatrix}.$$

Table 2 shows the number of iterations required for convergence of the GMRES method with different combinations of matrix sizes n and $u = 1/h$. In the table, $\mathcal{I}$ means no preconditioner is used and $\tilde{S}$ denotes the Strang–type BC preconditioner defined in (22). We see that the numbers of iterations required for convergence increase slowly for increasing n and u under the column $\tilde{S}$.

n	u	$\mathcal{I}$	$\tilde{S}$	n	u	$\mathcal{I}$	$\tilde{S}$
24	10	52	9	48	10	53	12
	20	97	11		20	98	14
	40	185	15		40	189	14
	80	367	19		80	378	17

Table 2: Number of iterations for convergence.

5 Singular Perturbation Delay Differential Equations

In this section, we study the solution of singular perturbation delay differential equations:

$$\begin{cases} \mathbf{x}'(t) = V^{(1)}\mathbf{x}(t) + V^{(2)}\mathbf{x}(t-\tau) + C^{(1)}\mathbf{y}(t) + C^{(2)}\mathbf{y}(t-\tau), & t \geq t_0, \\ \epsilon\mathbf{y}'(t) = F^{(1)}\mathbf{x}(t) + F^{(2)}\mathbf{x}(t-\tau) + G^{(1)}\mathbf{y}(t) + G^{(2)}\mathbf{y}(t-\tau), & t \geq t_0, \quad 0 < \epsilon \ll 1, \\ \mathbf{x}(t) = \boldsymbol{\phi}(t), & t \leq t_0, \\ \mathbf{y}(t) = \boldsymbol{\psi}(t), & t \leq t_0, \end{cases} \tag{23}$$

where $\mathbf{x}(t)$, $\boldsymbol{\phi}(t) : \mathbb{R} \to \mathbb{R}^m$; $\mathbf{y}(t)$, $\boldsymbol{\psi}(t) : \mathbb{R} \to \mathbb{R}^n$; $V^{(1)}$, $V^{(2)} \in \mathbb{R}^{m\times m}$; $C^{(1)}$, $C^{(2)} \in \mathbb{R}^{m\times n}$; $F^{(1)}$, $F^{(2)} \in \mathbb{R}^{n\times m}$; $G^{(1)}$, $G^{(2)} \in \mathbb{R}^{n\times n}$; and $\tau > 0$ is a constant. We can rewrite (23) as the following IVP,

$$\begin{cases} \mathbf{z}'(t) = P\mathbf{z}(t) + Q\mathbf{z}(t-\tau), & t \geq t_0, \\ \mathbf{z}(t) = \begin{pmatrix} \boldsymbol{\phi}(t) \\ \boldsymbol{\psi}(t) \end{pmatrix}, & t \leq t_0, \end{cases} \tag{24}$$

where P and $Q \in \mathbb{R}^{(m+n)\times(m+n)}$ are defined as follows,

$$P = \begin{pmatrix} V^{(1)} & C^{(1)} \\ \epsilon^{-1}F^{(1)} & \epsilon^{-1}G^{(1)} \end{pmatrix}, \qquad Q = \begin{pmatrix} V^{(2)} & C^{(2)} \\ \epsilon^{-1}F^{(2)} & \epsilon^{-1}G^{(2)} \end{pmatrix}.$$

see [12]. Let

$$h = \tau/k_2$$

be the step size where k_2 is a positive integer. For (24), by using a BVM with (ν_1, ν_2)-boundary conditions over a uniform mesh

$$t_j = t_0 + jh, \qquad j = 0, \cdots, v,$$

on the interval $[t_0, t_0 + vh]$, we have

$$\sum_{i=0}^{\mu} \alpha_i \mathbf{z}_{p+i-\nu_1} = h \sum_{i=0}^{\mu} \beta_i (P\mathbf{z}_{p+i-\nu_1} + Q\mathbf{z}_{p+i-\nu_1-k_2}), \tag{25}$$

for $p = \nu_1, \ldots, v-1$, where $\mu = \nu_1 + \nu_2$, and $\{\alpha_i\}_{i=0}^{\mu}$, $\{\beta_i\}_{i=0}^{\mu}$ are coefficients of the given BVM, see [5]. By providing the values

$$\mathbf{z}_{-k_2}, \ldots, \mathbf{z}_0, \quad \mathbf{z}_1, \ldots, \mathbf{z}_{\nu_1-1}, \quad \mathbf{z}_v, \ldots, \mathbf{z}_{v+\nu_2-1}, \tag{26}$$

(25) can be written in a matrix form as

$$K\mathbf{v} = \mathbf{b}, \tag{27}$$

where

$$K \equiv A \otimes I_{m+n} - hB \otimes P - hU \otimes Q. \tag{28}$$

The vector $\mathbf{v}$ in (27) is defined by

$$\mathbf{v}^T = (\mathbf{z}_{\nu_1}^T, \mathbf{z}_{\nu_1+1}^T, \ldots, \mathbf{z}_{v-1}^T) \in \mathbb{R}^{(m+n)(v-\nu_1)}.$$

The right-hand side $\mathbf{b} \in \mathbb{R}^{(m+n)(v-\nu_1)}$ of (27) depends on the boundary values and the coefficients of the method. The matrices $A, B \in \mathbb{R}^{(v-\nu_1)\times(v-\nu_1)}$ in (28) are defined as in (5) and $U \in \mathbb{R}^{(v-\nu_1)\times(v-\nu_1)}$ in (28) is defined as the matrix $B^{(1)}$ given by (12), see [5, 14, 6, 11].

The Strang-type BC preconditioner can be constructed for solving (27):

$$\hat{S} \equiv s(A) \otimes I_{m+n} - hs(B) \otimes P - hs(U) \otimes Q, \tag{29}$$

where $s(E)$ is Strang's circulant preconditioner of matrix E, for $E = A$, B, and U. We have the following theorem for the invertibility of our preconditioner $\hat{S}$ and for the convergence rate of our method.

Theorem 4 [12] *If the BVM with (ν_1, ν_2)-boundary conditions is A_{ν_1,ν_2}-stable and the conditions in Lemma 3 hold, i.e., $\eta(P) \equiv \frac{1}{2}\lambda_{\max}(P + P^T) < 0$ and $\eta(P) + ||Q||_2 < 0$, then the Strang-type preconditioner $\hat{S}$ defined by (29) is invertible. Moreover, when the GMRES method is applied to solving the preconditioned system*

$$\hat{S}^{-1}K\mathbf{v} = \hat{S}^{-1}\mathbf{b},$$

the method will converge in at most $\mathcal{O}(m+n)$ iterations in exact arithmetic.

We observe from Theorem 4 that if the step size $h = \tau/k_2$ is fixed, the number of iterations for convergence of the GMRES method applied to the system $\hat{S}^{-1}K\mathbf{v} = \hat{S}^{-1}\mathbf{b}$ will be independent of v, i.e., it is independent of the length of the interval we considered. For numerical examples, we refer to [12].

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