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Relations between the average bipartite entanglement and N-partite correlation functions

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Abstract

We study the relations between the average bipartite entanglement and the N-partite length of correlation. We show that the N-partite length of correlation can completely determine the average bipartite entanglement for two and three-qubits. For four-, five- and six-qubit systems, the N-partite correlation functions may not determine the average bipartite entanglement. These results are novel and promising for pure state.

Keywords: average bipartite entanglement, N-partite correlation functions, maximally multi-qubit entangled state

1. Introduction

Entanglement is considered as the central resource for quantum information and computation [1–4], and numerous theoretical and experimental works have been done in the field [5–8]. Since the last decade, a lot of efforts have been made to quantify the amount of entanglement of various multipartite states [9, 10]. In particular, investigations have been focused on maximally entangled states [11–13]. Bipartite entanglement is well understood but such characterization or classification in multiqubit states is still very challenging. On the other hand, when studying the entanglement between n particles, a natural extension is to consider N-partite correlations, i.e. the expectation value of the product of n measurement results. The holistic property of composite systems containing nonclassical correlations in their subsystems, has potential for many quantum processes [10]. For bipartite quantum systems, Schmidt decomposition allows single sum state vector under certain condition [14]. However, generally Schmidt decomposition does not exist for composite systems containing more than two subsystems [14]. Interestingly, existence of entanglement of pure states is fully captured by N-partite correlation functions [15–17]. A pure N-particle state is entangled if and only if the squared N-partite correlation functions averaged over uniform choices of local observables exceed a certain bound. In this letter, we investigate the relation between average bipartite entanglement and the N-partite length of correlation. We find that the N-partite length of correlation can completely determine the average bipartite entanglement for two and three-qubits. For four to six-qubit systems,

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the N-partite correlation functions cannot determine average bipartite entanglement.

2. The relations between the average bipartite entanglement and the N-partite correlation functions

In 2008, Facchi *et al* [11] proposed that the multipartite entanglement of a system of qubits can be characterized in terms of the distribution function of bipartite over all possible bipartitions of the qubits, namely

$$\pi_{\mathrm{ME}}=\begin{pmatrix} n \\ n_A \end{pmatrix}^{-1}\sum_{|A|=n_A}\pi_A, \tag{1}$$

where $n_A=[n/2]$, and the purity reads $\pi_A=\mathrm{Tr}_A\rho_A^2$, where $\rho_A=\mathrm{Tr}_{\bar{A}}|\psi\rangle\langle\psi|$ is the reduced density matrix of party A. The purity ranges between $\frac{1}{2^{n_A}}\leqslant\pi_A\leqslant 1$. The quantity π_{ME} in equation (1) measures the average bipartite entanglement over all possible balanced bipartitions. For a maximally multi-qubit entangled state, π_{ME} is minimal. In the following, we will investigate the relation between the average bipartite entanglement and the N-partite length of correlation.

2.1. Two-qubit pure states

We have

$$|\psi\rangle_{12}=a_0|00\rangle+a_1|01\rangle+a_2|10\rangle+a_3|11\rangle. \tag{2}$$

Then we have

$$\rho_{12}=|\psi\rangle_{12\,12}\langle\psi|. \tag{3}$$

It has been shown [13]

$$\mathrm{Tr}\rho_{12}^2=\frac{1}{4}+\frac{1}{4}(F_1+F_2+F_{12}) \tag{4}$$

$$\tau_2=\frac{1}{4}-\frac{1}{4}(F_1+F_2-F_{12}) \tag{5}$$

where

$\tau_2=|\langle\psi|\sigma_{1y}\otimes\sigma_{2y}|\psi^*\rangle|^2$, here the τ_2 is taken from [18],

$$F_i=\langle\psi|\hat{\sigma}_{ix}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iy}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iz}|\psi\rangle^2 \tag{6}$$

$$\begin{aligned}F_{ij}=&\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jz}|\psi\rangle^2\\&+\langle\psi|\hat{\sigma}_{iy}\hat{\sigma}_{jx}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iy}\hat{\sigma}_{jy}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iy}\hat{\sigma}_{jz}|\psi\rangle^2\\&+\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jx}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jy}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}|\psi\rangle^2.\end{aligned} \tag{7}$$

We have

$$\pi_{\mathrm{ME}}=\frac{1}{2}(\pi_1+\pi_2), \tag{8}$$

where $\pi_1=\mathrm{Tr}_1\rho_1^2=\frac{1}{2}+\frac{1}{2}F_1$, $\pi_2=\mathrm{Tr}_2\rho_2^2=\frac{1}{2}+\frac{1}{2}F_2$.

Then, we have

$$\pi_{\mathrm{ME}}=\frac{1}{2}+\frac{1}{4}(F_1+F_2). \tag{9}$$

Using (4)–(9), we have

$$\pi_{\mathrm{ME}}=\frac{1}{4}(5-F_{12}) \tag{10}$$

or

$$F_{12}=5-4\pi_{\mathrm{ME}}. \tag{11}$$

From (11), we know the length of correlation F_{12} can completely determine the average bipartite entanglement π_{ME} for two-qubit states.

For product state, we know that $\pi_{\mathrm{ME}}=1$, therefore $F_{12}=1$. For Bell states, one finds that $\pi_{\mathrm{ME}}=\frac{1}{2}$, therefore $F_{12}=3$. Hence the correlation functions arrive the maximally value [19].

2.2. Three-qubit systems

We have

$$\begin{aligned}|\psi\rangle_{123}=&a_0|000\rangle+a_1|001\rangle+a_2|010\rangle+a_3|011\rangle\\&+a_4|100\rangle+a_5|101\rangle+a_6|110\rangle+a_7|111\rangle.\end{aligned} \tag{12}$$

Then we have

$$\rho_{123}=|\psi\rangle_{123\,123}\langle\psi|. \tag{13}$$

Similarly, it has been shown that [13]

$$\mathrm{Tr}\rho_{123}^2=\frac{1}{8}+\frac{1}{8}(F_1+F_2+F_3+F_{12}+F_{13}+F_{23}+F_{123}) \tag{14}$$

$$0=\frac{1}{8}-\frac{1}{8}(F_1+F_2+F_3-F_{12}-F_{13}-F_{23}+F_{123}) \tag{15}$$

where

$$\begin{aligned}F_{ijk}=&\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{kx}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{ky}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{kz}|\psi\rangle^2\\&+\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}\hat{\sigma}_{kx}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}\hat{\sigma}_{ky}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}\hat{\sigma}_{kz}|\psi\rangle^2\\&+\cdots\\&+\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{kx}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{ky}|\psi\rangle^2+\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{kz}|\psi\rangle^2.\end{aligned} \tag{16}$$

The average bipartite entanglement can be expressed as

$$\pi_{\mathrm{ME}}=\frac{1}{3}(\pi_1+\pi_2+\pi_3). \tag{17}$$

Using (14)–(17), we have

$$\pi_{\mathrm{ME}}=\frac{1}{6}(7-F_{123}) \tag{18}$$

or

$$F_{123} = 7 - 6\pi_{\mathrm{ME}}. \tag{19}$$

From (19), we know that the length of correlation F_{123} can completely determine average bipartite entanglement π_{ME} for three-qubit states.

For product state, we know that $\pi_{\mathrm{ME}} = 1$, therefore $F_{123} = 1$. For Greenberger–Horne–Zeilinger (GHZ) states, one finds that: $\pi_{\mathrm{ME}} = \frac{1}{2}$, therefore $F_{123} = 4$. Hence again, correlation functions arrive their maximally value.

2.3. Four to six qubits systems

We have

$$\begin{aligned}|\psi\rangle_{1234} = {}& a_0|0000\rangle + a_1|0001\rangle + a_2|0010\rangle + a_3|0011\rangle \\ &+ a_4|0100\rangle + a_5|0101\rangle + a_6|0110\rangle + a_7|0111\rangle \\ &+ a_8|1000\rangle + a_9|1001\rangle + a_{10}|1010\rangle + a_{11}|1011\rangle \\ &+ a_{12}|1100\rangle + a_{13}|1101\rangle + a_{14}|1110\rangle + a_{15}|1111\rangle.\end{aligned} \tag{20}$$

Then, we have

$$\begin{aligned}\mathrm{Tr}\rho_{1234}^2 = {}& \frac{1}{16} + \frac{1}{16}(F_1 + F_2 + F_3 + F_4 + F_{12} \\ &+ F_{13} + F_{14} + F_{23} + F_{24} + F_{34} + F_{123} \\ &+ F_{124} + F_{134} + F_{234} + F_{1234})\end{aligned} \tag{21}$$

$$\begin{aligned}\tau_4 = {}& \frac{1}{16} - \frac{1}{16}(F_1 + F_2 + F_3 + F_4 \\ &- F_{12} - F_{13} - F_{14} - F_{23} - F_{24} - F_{34} \\ &+ F_{123} + F_{124} + F_{134} + F_{234} - F_{1234})\end{aligned} \tag{22}$$

where

$$\begin{aligned}F_{ijkl} = {}& \langle\psi|\hat\sigma_{ix}\hat\sigma_{jx}\hat\sigma_{kx}\hat\sigma_{lx}|\psi\rangle^2 + \langle\psi|\hat\sigma_{ix}\hat\sigma_{jx}\hat\sigma_{kx}\hat\sigma_{ly}|\psi\rangle^2 \\ &+ \langle\psi|\hat\sigma_{ix}\hat\sigma_{jx}\hat\sigma_{kx}\hat\sigma_{lz}|\psi\rangle^2 \\ &+ \langle\psi|\hat\sigma_{ix}\hat\sigma_{jx}\hat\sigma_{ky}\hat\sigma_{lx}|\psi\rangle^2 + \langle\psi|\hat\sigma_{ix}\hat\sigma_{jx}\hat\sigma_{ky}\hat\sigma_{ly}|\psi\rangle^2 \\ &+ \langle\psi|\hat\sigma_{ix}\hat\sigma_{jx}\hat\sigma_{ky}\hat\sigma_{lz}|\psi\rangle^2 + \cdots \\ &+ \langle\psi|\hat\sigma_{iz}\hat\sigma_{jz}\hat\sigma_{kz}\hat\sigma_{lx}|\psi\rangle^2 + \langle\psi|\hat\sigma_{iz}\hat\sigma_{jz}\hat\sigma_{kz}\hat\sigma_{ly}|\psi\rangle^2 \\ &+ \langle\psi|\hat\sigma_{iz}\hat\sigma_{jz}\hat\sigma_{kz}\hat\sigma_{lz}|\psi\rangle^2\end{aligned} \tag{23}$$

$$\tau_4 = |\langle\psi|\sigma_{1y}\otimes\sigma_{2y}\otimes\sigma_{3y}\otimes\sigma_{4y}|\psi^*\rangle|^2. \tag{24}$$

Here the τ_4 is taken from [18], and

$$\pi_{\mathrm{ME}} = \frac{1}{6}(\pi_{12} + \pi_{13} + \pi_{14} + \pi_{23} + \pi_{24} + \pi_{34}). \tag{25}$$

Using (21)–(24), we have

$$\pi_{\mathrm{ME}} = \frac{1}{6}\left(9\frac{1}{4} - 4\tau_4 - \frac{3}{4}(F_{123} + F_{124} + F_{134} + F_{234}) - \frac{1}{4}F_{1234}\right) \tag{26}$$

or

$$F_{1234} = 37 - 16\tau_4 - 3(F_{123} + F_{124} + F_{134} + F_{234}) - 24\pi_{\mathrm{ME}}. \tag{27}$$

From (26), we know for four-qubit systems, the correlation functions cannot determine the average bipartite entanglement.

For product state, we know that

$$\tau_4 = 0, \pi_{\mathrm{ME}} = 1, F_{123} = F_{124} = F_{134} = F_{234} = 1,$$

therefore $F_{1234} = 1$. For GHZ states, one finds that [19]:

$$\pi_{\mathrm{ME}} = \frac{1}{2}, \tau_4 = 1, F_{123} = F_{124} = F_{134} = F_{234} = 0$$

therefore $F_{1234} = 9$. For maximally entangled states, one finds that:

$$\pi_{\mathrm{ME}} = \frac{1}{3}, \tau_4 = 0, F_{123} + F_{124} + F_{134} + F_{234} = 8,$$

therefore $F_{1234} = 5$.

For five-qubits, similarly, we have

$$\begin{aligned}\mathrm{Tr}\rho_{12345}^2 = {}& \frac{1}{32} + \frac{1}{32}(F_1 + F_2 + F_3 + F_4 + F_5 \\ &+ F_{12} + F_{13} + F_{14} + F_{15} + F_{23} + F_{24} + F_{25} + F_{34} \\ &+ F_{35} + F_{45} + F_{123} + F_{124} + F_{125} + F_{134} + F_{135} \\ &+ F_{145} + F_{234} + F_{235} + F_{345} + F_{1234} + F_{1235} \\ &+ F_{1245} + F_{1345} + F_{2345} + F_{12345})\end{aligned}$$

$$\begin{aligned}= {}& \frac{1}{32} - \frac{1}{32}(F_1 + F_2 + F_3 + F_4 + F_5 \\ &- F_{12} - F_{13} - F_{14} - F_{15} - F_{23} - F_{24} - F_{25} - F_{34} \\ &- F_{35} - F_{45} + F_{123} + F_{124} + F_{125} + F_{134} + F_{135} + F_{145} \\ &+ F_{234} + F_{235} + F_{345} - F_{1234} - F_{1235} \\ &- F_{1245} - F_{1345} - F_{2345} + F_{12345})\end{aligned}$$

$$\begin{aligned}\pi_{\mathrm{ME}} = {}& \frac{1}{10}(\pi_{12} + \pi_{13} + \pi_{14} + \pi_{15} + \pi_{23} + \pi_{24} \\ &+ \pi_{25} + \pi_{34} + \pi_{35} + \pi_{45}).\end{aligned}$$

Then we have

$$\pi_{\mathrm{ME}}=\frac{1}{10}\begin{pmatrix} 22\frac{1}{4}-(F_{123}+F_{124}+F_{125}+F_{134}+F_{135}+F_{145}+F_{234}+F_{235}+F_{245}+F_{345}) \\ -\frac{1}{4}(F_{1234}+F_{1235}+F_{1245}+F_{1345}+F_{2345})-F_{12345} \end{pmatrix} \quad (28)$$

or,

$$F_{12345}=22\frac{1}{4}-(F_{123}+F_{124}+F_{125}+F_{134}+F_{135}+F_{145}+F_{234}+F_{235}+F_{245}+F_{345})-\frac{1}{4}(F_{1234}+F_{1235}+F_{1245}+F_{1345}+F_{2345})-10\pi_{\mathrm{ME}}. \quad (29)$$

For product state,

$$\pi_{\mathrm{ME}}=1, F_{123}=F_{124}=F_{134}=F_{234}=1, F_{1234}=1, F_{12345}=1.$$

For GHZ states, one finds:

$$\pi_{\mathrm{ME}}=\frac{1}{2}, F_{123}=F_{124}=F_{134}=F_{234}=0., F_{1234}=F_{1235}=F_{1245}=F_{1345}=F_{2345}=1, F_{12345}=16.$$

For maximally entangled states, one finds:

$$\pi_{\mathrm{ME}}=\frac{1}{4}, F_{123}=F_{124}=\cdots=F_{345}=1, F_{1234}=F_{1235}=F_{1245}=F_{1345}=F_{2345}=3, F_{12345}=6.$$

For $n=6$ qubits, one can show that

$$\mathrm{Tr}\rho^2_{123456}=\frac{1}{64}+\frac{1}{64}(F_1+F_2+F_3+F_4+F_5+F_6+F_{12}+F_{13}+F_{14}+\cdots+F_{56}+F_{123}+F_{124}+F_{125}+\cdots+F_{456}+F_{1234}+F_{1235}+\cdots+F_{3456}+F_{12345}+F_{12346}+\cdots+F_{23456}+F_{123456})$$

$$\tau_6=\frac{1}{64}-\frac{1}{64}(F_1+F_2+F_3+F_4+F_5+F_6-F_{12}-F_{13}-F_{14}-\cdots-F_{56}+F_{123}+F_{124}+F_{125}+\cdots+F_{456}-F_{1234}-F_{1235}-\cdots-F_{3456}+F_{12345}+F_{12346}+\cdots+F_{23456}-F_{123456})$$

$$\tau_6=|\langle\psi|\sigma_{1y}\otimes\sigma_{2y}\otimes\sigma_{3y}\otimes\sigma_{4y}\otimes\sigma_{5y}\otimes\sigma_{6y}|\psi^*\rangle|^2. \quad (30)$$

Then, we have

$$\pi_{\mathrm{ME}}=\frac{1}{20}\begin{pmatrix} 31+3\tau_6-\frac{1}{2}(F_{1234}+F_{1235}+\cdots+F_{3456}) \\ -\frac{1}{2}(F_{12345}+F_{12346}+\cdots+F_{23456})-\frac{1}{2}F_{123456} \end{pmatrix} \quad (31)$$

or,

$$F_{123456}=62+6\tau_6-(F_{1234}+F_{1235}+\cdots+F_{3456})-(F_{12345}+F_{12346}+\cdots+F_{23456})-40\pi_{\mathrm{ME}}. \quad (32)$$

For product state,

$$\tau_6=0, F_{1234}+F_{1235}+\cdots+F_{3456}=1, F_{12345}+F_{12346}+\cdots+F_{23456}=1, \pi_{\mathrm{ME}}=1, F_{123456}=1.$$

For GHZ states, one finds:

$$\tau_6=1, F_{1234}+F_{1235}+\cdots+F_{3456}=1, F_{12345}+F_{12346}+\cdots+F_{23456}=0, \pi_{\mathrm{ME}}=\frac{1}{2}, F_{123456}=33.$$

For the maximally entangled states, one finds:

$$\tau_6=1, F_{1234}+F_{1235}+\cdots+F_{3456}=3, F_{12345}+F_{12346}+\cdots+F_{23456}=0, \pi_{\mathrm{ME}}=\frac{1}{8}, F_{123456}=18.$$

From equations (27), (29) and (32), the correlation function does not depend on the average bipartite entanglement uniquely for four, five and six qubit states.

3. Conclusion

In summary, we introduce a relationship between the average bipartite entanglement and the N-partite length of correlation. We show that the N-partite length of the correlation can completely determine the average bipartite entanglement for two and three-qubit. For four, five, six-qubit GHZ states, we find that the N-partite correlation functions are larger than those of maximally entangled states. These results are novel and promising for pure state.

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